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## The Algebraic Degree of Semidefinite Programming

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**Abstract.** Given a generic semidefinite program, specified by matrices with rational entries, each coordinate of its optimal solution is an algebraic number. We study the degree of the minimal polynomials of these algebraic numbers. Geometrically, this degree counts the critical points attained by a linear functional on a fixed rank locus in a linear space of symmetric matrices. We determine this degree using methods from complex algebraic geometry, such as projective duality, determinantal varieties, and their Chern classes.

**Key words.** Semidefinite programming, algebraic degree, genericity, determinantal variety, dual variety, multidegree, Euler-Poincaré characteristic, Chern class

### 1. Introduction

A fundamental question about any optimization problem is how the output depends on the input. The set of optimal solutions and the optimal value of the problem are functions of the parameters, and it is important to understand the nature of these functions. For instance, for a linear programming problem,

$$\text{maximize } c \cdot x \text{ subject to } A \cdot x = b \text{ and } x \geq 0, \quad (1.1)$$

the optimal value is convex and piecewise linear in the cost vector  $c$  and the right hand side  $b$ , and it is a piecewise rational function of the entries of the matrix  $A$ . The area of mathematics which studies these functions is geometric combinatorics, specifically the theory of matroids for the dependence on  $A$ , and the theory of polyhedral subdivisions [5] for the dependence on  $b$  and  $c$ .

For a second example, consider the following basic question in game theory:

$$\text{Given a game, compute its Nash equilibria.} \quad (1.2)$$

If there are only two players and one is interested in fully mixed Nash equilibria then this is a linear problem, and in fact closely related to linear programming. On the other hand, if the number of players is more than two then the problem (1.2) is universal in the sense of real algebraic geometry: Datta [4] showed that every real algebraic variety is isomorphic to the set of Nash equilibria of some three-person game. A corollary of her construction is that, if the Nash equilibria are discrete, then their coordinates can be arbitrary algebraic functions of the given input data (the payoff values which specify the game).

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Our third example concerns maximum likelihood estimation in statistical models for discrete data. Here the optimization problem is as follows:

$$\text{Maximize } p_1(\theta)^{u_1} p_2(\theta)^{u_2} \cdots p_n(\theta)^{u_n} \text{ subject to } \theta \in \Theta, \quad (1.3)$$

where  $\Theta$  is an open subset of  $\mathbb{R}^m$ , the  $p_i(\theta)$  are polynomial functions that sum to one, and the  $u_i$  are positive integers (these are the data). The optimal solution  $\hat{\theta}$ , which is the maximum likelihood estimator, depends on the data:

$$(u_1, \dots, u_n) \mapsto \hat{\theta}(u_1, \dots, u_n). \quad (1.4)$$

This is an algebraic function, and recent work in algebraic statistics [3, 12] has led to a formula for the degree of this algebraic function, under certain hypothesis on the polynomials  $p_i(\theta)$  which specify the statistical model.

The aim of the present paper is to conduct a similar algebraic analysis for the optimal value function in *semidefinite programming*. This function shares some key features with each of the previous examples. To begin with, it is a convex function which is piecewise algebraic. However, unlike for (1.1), the pieces are non-linear, so there is a notion of algebraic degree as for Nash equilibria (1.2) and maximum likelihood estimation (1.3). However, semidefinite programming does not exhibit universality as in [4] because the structure of real symmetric matrices imposes some serious constraints on the underlying algebraic geometry. It is these constraints we wish to explore and characterize.

We consider the semidefinite programming (SDP) problem in the form

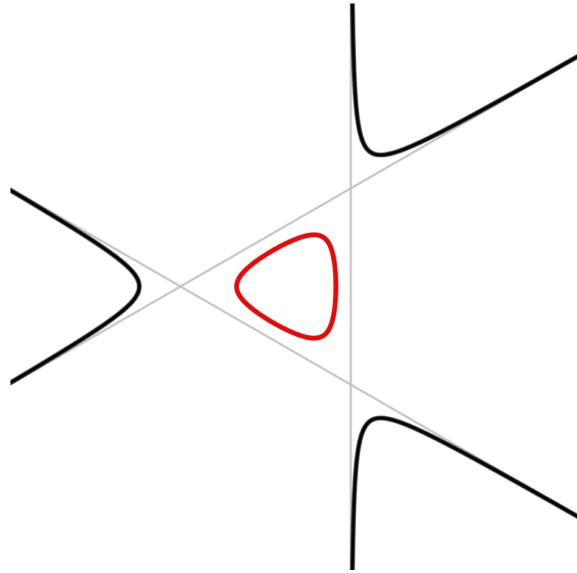
$$\text{maximize } \text{trace}(B \cdot Y) \text{ subject to } Y \in \mathcal{U} \text{ and } Y \succeq 0. \quad (1.5)$$

where  $B$  is a real symmetric  $n \times n$ -matrix,  $\mathcal{U}$  is a  $m$ -dimensional affine subspace in the  $\binom{n+1}{2}$ -dimensional space of real  $n \times n$ -symmetric matrices, and  $Y \succeq 0$  means that  $Y$  is positive semidefinite (all  $n$  eigenvalues of  $Y$  are non-negative). The problem (1.5) is *feasible* if and only if the subspace  $\mathcal{U}$  intersects the cone of positive semidefinite matrices. In the range where it exists and is unique, the optimal solution  $\hat{Y}$  of this problem is a piecewise algebraic function of the matrix  $B$  and the subspace  $\mathcal{U}$ . Our aim is to understand the geometry of this function.

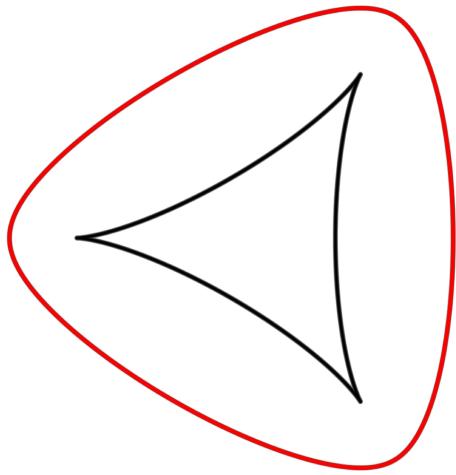
Note if  $\mathcal{U}$  consists of diagonal matrices only, then we are in the linear programming case (1.1), and the algebraic degree of the pieces of  $\hat{Y}$  is just one. What we are interested in is the increase in algebraic complexity which arises from the passage from diagonal matrices to non-diagonal symmetric matrices.

**Example 1 (Elliptic Vinnikov curves)** Let  $n = 3$  and  $m = 2$ , so  $Y$  runs over a two-dimensional affine space of symmetric  $3 \times 3$ -matrices. Then  $Y \succeq 0$  specifies a closed convex semi-algebraic region in this plane, whose boundary is the central connected component of a cubic curve as depicted in Figure 1. This curve is a Vinnikov curve, which means that it satisfies the following constraint in real algebraic geometry: any line which meets the interior of the convex region intersects this cubic curve in three real points. See [11, 15, 20, 24] for details. However, there are no constraints on Vinnikov curves in the setting of complex algebraic geometry. The Vinnikov constraints involve inequalities and no equations. This explains why the curve in Figure 1 is smooth.

Our problem (1.5) is to maximize a linear function over the convex component. Algebraically, the restriction of a linear function to the cubic curve has six critical points, two of which are complex. They correspond to the intersection points of the dual curve with a line in the dual projective plane. The degree six curve dual to the elliptic Vinnikov curve is depicted in Figure 2.

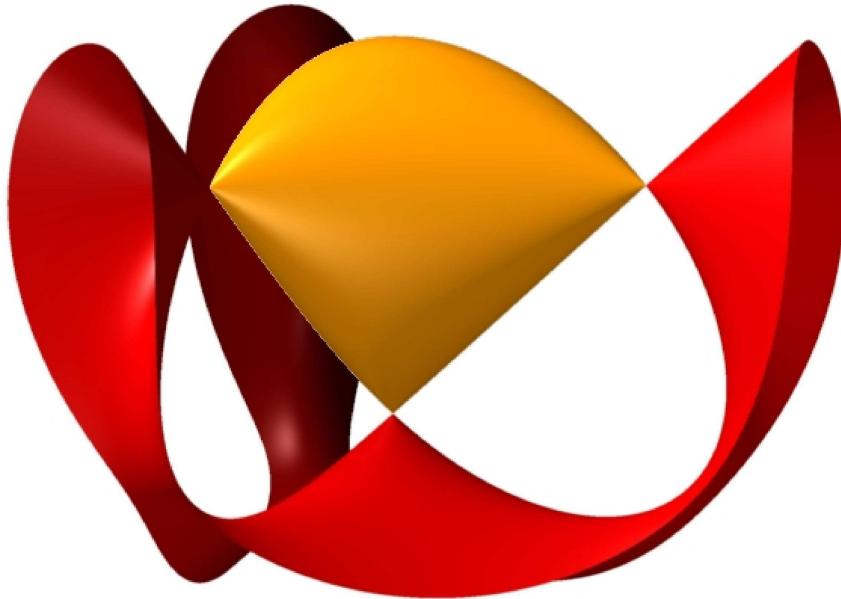


**Fig. 1.** The convex component in the center of this elliptic Vinnikov curve is the feasible region for SDP with  $m = 2, n = 3$ .



**Fig. 2.** The dual to the elliptic Vinnikov curve in Figure 1 is a plane curve of degree six with three real singular points.

Our analysis shows that the algebraic degree of SDP equals six when  $m = 2$  and  $n = 3$ . If the matrix  $B$  and the plane  $\mathcal{U}$  are defined over  $\mathbb{Q}$  then the coordinates of the optimal solution  $\hat{Y}$  are algebraic numbers of degree six. By Galois theory, the solution  $\hat{Y}$  cannot in general be expressed in terms of radicals. For any specific numerical instance we can use the command “galois” in **maple** to compute the Galois group, which is then typically found to be the symmetric group  $S_6$ .  $\square$



**Fig. 3.** The convex component in the center of this Cayley cubic surface is the feasible region for SDP with  $m = n = 3$ .

**Example 2 (The Cayley Cubic)** Now, suppose that  $m = n = 3$ . Then  $\det(Y) = 0$  is a cubic surface, but this surface is constrained in the context of complex algebraic geometry because it cannot be smooth. The cubic surface  $\det(Y) = 0$  has four isolated nodal singularities, namely, the points where  $X$  has rank one. This cubic surface is known to geometers as the Cayley cubic. In the optimization literature, it occurs under the names elliptope or symmetroid. Optimization experts are familiar with (the convex component of) the Cayley cubic surface from (the upper left hand picture in) Christoph Helmberg's SDP web page <http://www-user.tu-chemnitz.de/~helmberg/semidef.html>.

The surface dual to the Cayley cubic is a surface of degree four, which is known as the Steiner surface. There are now two possibilities for the optimal solution  $\hat{Y}$  of (1.5). Either  $\hat{Y}$  has rank one, in which case it is one of the four singular points of the cubic surface in Figure 3, or  $\hat{Y}$  has rank two and is gotten by intersecting the Steiner surface by a line specified by  $B$ . In either of these two cases, the optimal solution  $\hat{Y}$  is an algebraic function of degree four in the data specifying  $B$  and  $\mathcal{U}$ . In particular, using Girolamo Cardano's *Ars Magna*, we can express the coordinates of  $\hat{Y}$  in terms of radicals in  $(B, \mathcal{U})$ .  $\square$

The objective of this paper is to study the geometric figures shown in Figures 1, 2 and 3 for arbitrary values of  $n$  and  $m$ . The targeted audience includes both algebraic geometers and scholars in optimization. Our presentation is organized as follows. In Section 2 we review SDP duality, we give an elementary introduction to the notion of algebraic degree, and we explain what it means for the data  $\mathcal{U}$  and  $B$  to be generic. In Section 3 we derive Pataki's inequalities which characterize the possible ranks of the optimal matrix  $\hat{Y}$ , and, in Theorem 7, we present a precise characterization of the algebraic degree. The resulting geometric formulation of semidefinite programming is our point of departure in Section 4. Theorem 10 expresses the algebraic degree of SDP as a certain bidegree. This is a notion of degree for subvarieties of products of projective spaces, and is an instance of the general definition of multidegree in Section 8.5 of the text book [17]. Theorem 11 gives explicit formulas for the

algebraic degree, organized according to the rows of Table 2. In Section 5 we present results involving projective duality and determinantal varieties. in Section 6 this is combined with results of Pragacz [19] to prove Theorem 11, and to derive the general formula stated in Theorem 19 and Conjecture 21.

Two decades ago, the concept of algebraic degree of an optimization problem had been explored in the computational geometry literature, notably in the work of Bajaj [2]. However, that line of research had only few applications, possibly because of the dearth of precise results for geometric problems of interest. Our paper fills this gap, at least for problems with semidefinite representation, and it can be read as an invitation to experts in complexity theory to take a fresh look at Bajaj’s conclusion that “... *the domain of relations between the algebraic degree ... and the complexity of obtaining the solution point of optimization problems is an exciting area to explore.*” [2, page 190].

The algebraic degree of semidefinite programming addresses the computational complexity at a fundamental level. To solve the semidefinite programming exactly essentially reduces to solve a class of univariate polynomial equations whose degrees are the algebraic degree. As we will see later in this paper, the algebraic degree is usually very big, even for some small problems. An explicit general formula for the algebraic degree was given by von Bothmer and Ranestad in the paper [25] which was written subsequently to this article.

## 2. Semidefinite programming: duality and symbolic solutions

In this section we review the duality theory of semidefinite programming, and we give an elementary introduction to the notion of algebraic degree.

Let  $\mathbb{R}$  be the field of real numbers and  $\mathbb{Q}$  the subfield of rational numbers. We write  $\mathcal{S}^n$  for the  $\binom{n+1}{2}$ -dimensional vector space of symmetric  $n \times n$ -matrices over  $\mathbb{R}$ , and  $\mathbb{Q}\mathcal{S}^n$  when we only allow entries in  $\mathbb{Q}$ . A matrix  $X \in \mathcal{S}^n$  is *positive definite*, denoted  $X \succ 0$ , if  $u^T X u > 0$  for all  $u \in \mathbb{R}^n \setminus \{0\}$ , and it is *positive semidefinite*, denoted  $X \succeq 0$ , if  $u^T X u \geq 0$  for all  $u \in \mathbb{R}^n$ . We consider the *semidefinite programming* (SDP) problem

$$\min_{X \in \mathcal{S}^n} C \bullet X \tag{2.1}$$

$$\text{s.t. } A_i \bullet X = b_i \quad \text{for } i = 1, \dots, m \tag{2.2}$$

$$\text{and } X \succeq 0 \tag{2.3}$$

where  $b \in \mathbb{Q}^m$ ,  $C, A_1, \dots, A_m \in \mathbb{Q}\mathcal{S}^n$ . The inner product  $C \bullet X$  is defined as

$$C \bullet X := \text{trace}(C \cdot X) = \sum C_{ij} X_{ij} \quad \text{for } C, X \in \mathcal{S}^n.$$

This is a linear function in  $X$  for fixed  $C$ , and that is our *objective function*. The primal SDP problem (2.2)-(2.3) is called *strictly feasible*, or the feasible region *has an interior point*, if there exists some  $X \succ 0$  such that (2.2) is met.

Throughout this paper, the words “generic” and “genericity” appear frequently. These notions have a precise meaning in algebraic geometry: the data  $C, b, A_1, \dots, A_m$  are *generic* if their coordinates satisfy no non-zero polynomial equation with coefficients in  $\mathbb{Q}$ . Any statement that is proved under such a genericity hypothesis will be valid for all data that lie in a dense, open subset of the space of data, and hence it will hold except on a set of Lebesgue measure zero. For a simple illustration consider the quadratic equation  $\alpha t^2 + \beta t + \gamma = 0$  where  $t$  is the variable and  $\alpha, \beta, \gamma$  are certain parameters. This equation has two distinct roots if and only if the discriminant  $\alpha(\beta^2 - 4\alpha\gamma)$  is non-zero. The equation  $\alpha(\beta^2 - 4\alpha\gamma) = 0$  defines a surface, which has measure zero in 3-space. The general point  $(\alpha, \beta, \gamma)$  does not lie

on this surface. So we can say that  $\alpha t^2 + \beta t + \gamma = 0$  has two distinct roots when  $\alpha, \beta, \gamma$  are generic.

The convex optimization problem dual to (2.1)-(2.2) is as follows:

$$\max_{y \in \mathbb{R}^m} b^T y \quad (2.4)$$

$$\text{s.t. } A(y) := C - \sum_{i=1}^m y_i A_i \succeq 0. \quad (2.5)$$

Here the decision variables  $y_1, \dots, y_m$  are real unknown. The condition (2.5) is also called a *linear matrix inequality* (LMI). We say that (2.5) is *strictly feasible* or has an *interior point* if there exists some  $y \in \mathbb{R}^m$  such that  $A(y) \succ 0$ .

Our formulation of semidefinite programming in (1.5) is equivalent to (2.4)–(2.5) under the following identifications. Take  $\mathcal{U}$  to be the affine space consisting of all matrices  $C - \sum_{i=1}^m y_i A_i$  where  $y \in \mathbb{R}^m$ , write  $Y = A(y)$  for an unknown matrix in this space, and fix a matrix  $B$  such that  $B \bullet A_i = -b_i$  for  $i = 1, \dots, m$ . Such a choice is possible provided the matrices  $A_i$  are chosen to be linearly independent, and it implies  $B \bullet Y - B \bullet C = b^T y$ .

We refer to [23, 26] for the theory, algorithms and applications of SDP. The known results concerning SDP duality can be summarized as follows. Suppose that  $X \in \mathcal{S}^n$  is feasible for (2.2)–(2.3) and  $y \in \mathbb{R}^m$  is feasible for (2.5). Then  $C \bullet X - b^T y = A(y) \bullet X \geq 0$ , because the inner product of any two semidefinite matrices is non-negative. Hence the following *weak duality* always holds:

$$\sup_{A(y) \succeq 0} b^T y \leq \inf_{\substack{X \succeq 0 \\ \forall i : A_i \bullet X = b_i}} C \bullet X.$$

When equality holds in this inequality, then we say that *strong duality* holds.

The cone of positive semidefinite matrices is a self-dual cone. This has the following important consequence for any positive semidefinite matrices  $A(y)$  and  $X$  which represent feasible solutions for (2.2)–(2.3) and (2.5). The inner product  $A(y) \bullet X$  is zero if and only if the matrix product  $A(y) \cdot X$  is the zero matrix. The optimality conditions are summarized in the following theorem:

**Theorem 3 (Section 3 in [23] or Chapter 4 in [26]).** *Suppose that both the primal problem (2.1)–(2.3) and the dual problem (2.4)–(2.5) are strictly feasible. Then strong duality holds, there exists a pair of optimal solutions, and the following optimality conditions characterize the pair of optimal solutions:*

$$A_i \bullet \hat{X} = b_i \quad \text{for } i = 1, 2, \dots, m, \quad (2.6)$$

$$A(\hat{y}) \cdot \hat{X} = 0, \quad (2.7)$$

$$A(\hat{y}) \succeq 0 \text{ and } \hat{X} \succeq 0. \quad (2.8)$$

The matrix equation (2.7) is the *complementarity condition*. It implies that the sum of the ranks of the matrices  $A(\hat{y})$  and  $\hat{X}$  is at most  $n$ . We say that *strict complementarity* holds if the sum of the ranks of  $A(\hat{y})$  and  $\hat{X}$  equals  $n$ .

Suppose now that the given data  $C, A_1, \dots, A_m$  and  $b$  are generic over the rational numbers  $\mathbb{Q}$ . In practice, we may choose the entries of these matrices to be random integers, and we compute the optimal solutions  $\hat{y}$  and  $\hat{X}$  from these data, using a numerical interior point method. Our objective is to learn by means of algebra how  $\hat{y}$  and  $\hat{X}$  depend on the input  $C, A_1, \dots, A_m, b$ . Our approach rests on the optimality conditions in Theorem 3. These take the form of a system of polynomial equations. If we could solve these equations using symbolic computation, then this would furnish an exact representation of the optimal solution  $(\hat{X}, \hat{y})$ . We illustrate this approach for a small example.

**Example 4** Consider the following semidefinite programming problem:

$$\begin{aligned} \text{Maximize } & y_1 + y_2 + y_3 \\ \text{subject to } A(y) := & \begin{bmatrix} y_3 + 1 & y_1 + y_2 & y_2 & y_2 + y_3 \\ y_1 + y_2 & -y_1 + 1 & y_2 - y_3 & y_2 \\ y_2 & y_2 - y_3 & y_2 + 1 & y_1 + y_3 \\ y_2 + y_3 & y_2 & y_1 + y_3 - y_3 + 1 & \end{bmatrix} \succeq 0. \end{aligned}$$

This is an instance of the LMI formulation (2.4)-(2.5) with  $m = 3$  and  $n = 4$ . Using the numerical software **SeDuMi** [21], we easily find the optimal solution:

$$(\hat{y}_1, \hat{y}_2, \hat{y}_3) = (0.3377\ldots, 0.5724\ldots, 0.3254\ldots).$$

What we are interested in is to understand the nature of these three numbers.

To examine this using symbolic computation, we also consider the primal formulation (2.1)–(2.3). Here the decision variable is a symmetric  $4 \times 4$ -matrix  $X = (x_{ij})$  whose ten entries  $x_{ij}$  satisfy the three linear constraints (2.6):

$$\begin{aligned} A_1 \bullet X &= -2x_{12} + x_{22} - 2x_{34} = 1 \\ A_2 \bullet X &= -2x_{12} - 2x_{13} - 2x_{14} - 2x_{23} - 2x_{24} - x_{33} = 1 \\ A_3 \bullet X &= x_{11} - 2x_{14} + 2x_{23} - 2x_{34} + x_{44} = 1 \end{aligned}$$

In addition, there are sixteen quadratic constraints coming from the complementarity condition (2.7), namely we equate each entry of the matrix product  $A(y) \cdot X$  with zero. Thus (2.6)–(2.7) translates into a system of 19 linear and quadratic polynomial equations in the 13 unknowns  $y_1, y_2, y_3, x_{11}, x_{12}, \dots, x_{44}$ .

Using symbolic computation methods (namely, Gröbner bases in **Macaulay 2** [8]), we find that these equations have finitely many complex solutions. The number of solutions is 26. Indeed, by eliminating variables, we discover that each coordinate  $y_i$  or  $x_{jk}$  satisfies a univariate equation of degree 26. Interestingly, these univariate polynomials are not irreducible but they factor. For instance, the optimal first coordinate  $\hat{y}_1$  satisfies the univariate polynomial  $f(y_1)$  which factors into a polynomial  $g(y_1)$  of degree 16 and a polynomial of degree  $h(y_1)$  of degree 10. Namely, we have  $f(y_1) = g(y_1) \cdot h(y_1)$ , where

$$\begin{aligned} g(y_1) = & 403538653715069011 y_1^{16} - 2480774864948860304 y_1^{15} \\ & + 6231483282173647552 y_1^{14} - 5986611777955575152 y_1^{13} \\ & + \dots \dots \dots \dots + \\ & + 59396088648011456 y_1^2 - 4451473629111296 y_1 + 149571632340416 \end{aligned}$$

and  $h(y_1) = 2018 y_1^{10} - 12156 y_1^9 + 17811 y_1^8 + \dots + 1669 y_1 - 163$ .

Both of these polynomials are irreducible in  $\mathbb{Q}[y_1]$ . By plugging in, we see that the optimal first coordinate  $\hat{y}_1 = 0.3377\ldots$  satisfies  $g(\hat{y}_1) = 0$ . Hence  $\hat{y}_1$  is an algebraic number of degree 16 over  $\mathbb{Q}$ . Indeed, each of the other twelve optimal coordinates  $\hat{y}_2, \hat{y}_3, \hat{x}_{11}, \hat{x}_{12}, \dots, \hat{x}_{44}$  also has degree 16 over  $\mathbb{Q}$ . We conclude that the algebraic degree of this particular SDP problem is 16. Note that the optimal matrix  $A(\hat{y})$  has rank 3 and the matrix  $\hat{X}$  has rank 1.

We are now in a position to vary the input data and perform a parametric analysis. For instance, if the objective function is changed to  $y_1 - y_2$  then the algebraic degree is 10 and the ranks of the optimal matrices are both 2.  $\square$

The above example is not special. The entries for  $m = 3, n = 4$  in Table 2 below inform us that, for generic data  $(C, b, A_i)$ , the algebraic degree is 16 when the optimal matrix  $\hat{Y} = A(\hat{y})$  has rank three, the algebraic degree is 10 when the optimal matrix  $\hat{Y} = A(\hat{y})$  has rank two, and rank one or four optimal solutions do not exist. These former two cases can be understood geometrically by drawing a picture as in Figure 3 and Example 2. The surface  $\det(Y) = 0$  has degree four and it has 10 isolated singular points (these are the matrices  $Y$  of rank two). The surface dual to the quartic  $\det(Y) = 0$  has degree 16. Our optimal solution in Example 4 is one of the 16 intersection points of this dual surface with the line specified by the linear objective function  $B \bullet Y = b \cdot y^T$ . The concept of duality in algebraic geometry will be reviewed in Section 5.

For larger semidefinite programming problems it is impossible (and hardly desirable) to solve the polynomial equations (2.6)–(2.7) symbolically. However, we know that the coordinates  $\hat{y}_i$  and  $\hat{x}_{jk}$  of the optimal solution are the roots of some univariate polynomials which are irreducible over  $\mathbb{Q}$ . If the data are generic, then the degree of these univariate polynomials depends only on the rank of the optimal solution. This is what we call the *algebraic degree* of the semidefinite programming problem (2.1)–(2.3) and its dual (2.4)–(2.5). The objective of this paper is to find a formula for this algebraic degree.

### 3. From Pataki's inequalities to algebraic geometry

Example 4 raises the question which ranks are to be expected for the optimal matrices. This question is answered by the following result from the SDP literature [1, 18]. We refer to the semidefinite program specified in (2.1)–(2.3), (2.4)–(2.5) and (2.6)–(2.8). Furthermore, we always assume that the problem instance  $(C, b, A_1, \dots, A_m)$  is generic, in the sense discussed above.

**Proposition 5** (Pataki's Inequalities [18, Corollary 3.3.4], see also [1])  
*Let  $r$  and  $n - r$  be the ranks of the optimal matrices  $\hat{Y} = A(\hat{y})$  and  $\hat{X}$ . Then*

$$\binom{n-r+1}{2} \leq m \quad \text{and} \quad \binom{r+1}{2} \leq \binom{n+1}{2} - m. \quad (3.1)$$

A proof will be furnished later in this section. First we illustrate Proposition 5 by describing a numerical experiment which is easily performed for a range of values  $(m, n)$ . We generated  $m$ -tuples of  $n \times n$ -matrices  $(A_1, A_2, \dots, A_m)$ , where each entry was independently drawn from the standard Gaussian distribution. Then, for a random positive semidefinite matrix  $X_0$ , let  $b_i = A_i \bullet X_0$ , which makes the feasible set (2.2)–(2.3) nonempty. For each such choice, we generated 10000 random symmetric matrices  $C$ . Using SeDuMi [21] we then solved the program (2.1)–(2.3) and we determined the numerical rank of its optimal solution  $\hat{Y}$ . This was done by computing the Singular Value Decomposition of  $\hat{Y}$  in Matlab. The result is the rank distribution in Table 1.

Table 1 verifies that, with probability one, the rank  $r$  of the optimal matrix  $\hat{Y}$  lies in the interval specified by Pataki's Inequalities. The case  $m = 2$ , which concerns Vinnikov curves as in Example 1, is not listed in Table 1 because here the optimal rank is always  $n - 1$ . The first interesting case is  $m = n = 3$ , which concerns Cayley's cubic surface as in Example 2. When optimizing a linear function over the convex surface in Figure 3, it is three times less likely for a smooth point to be optimal than one of the four vertices. For  $m = 3, n = 4$ , as in Example 4, the odds are slightly more balanced. In only 35.34% of the instances the algebraic degree of the optimal solution was 16, while in 64.66% of the instances the algebraic degree was found to be 10.

$n$	3		4		5		6	
$m$	rank	percent	rank	percent	rank	percent	rank	percent
3	2	24.00%	3	35.34%	4	79.18%	5	82.78%
	1	76.00%	2	64.66%	3	20.82%	4	17.22%
4			3	23.22%	4	16.96%	5	37.42%
	1	100.00%	2	76.78%	3	83.04%	4	62.58%
5					4	5.90%	5	38.42%
	1	100.00%	2	100.00 %	3	94.10%	4	61.58%
6			2	67.24%	3	93.50%	5	1.32%
			1	32.76%	2	6.50%	4	93.36%
			2	52.94%	3	82.64%	3	5.32%
7			1	47.06%	2	17.36%	4	78.82%
					3	34.64%	3	21.18%
8			1	100.00%	2	65.36%	4	45.62%
					3	7.60%	3	54.38%
9			1	100.00%	2	92.40%	4	23.50%
					3	76.50%		

**Table 1.** Distribution of the rank of the optimal matrix  $\hat{Y}$ .

This experiment highlights again the genericity hypothesis made throughout this paper. We shall always assume that the  $m$ -tuple  $(A_1, \dots, A_m)$ , the cost matrix  $C$  and vector  $b$  are generic. All results in this paper are only valid under this genericity hypothesis. Naturally, special phenomena will happen for special problem instances. For instance, the rank of the optimal matrix can be outside the Pataki interval. While such special instances are important for applications of SDP, we shall not address them in this present study.

In what follows we introduce our algebraic setting. We fix the assumptions

$$b_1 = 1 \text{ and } b_2 = b_3 = \dots = b_m = 0. \quad (3.2)$$

These assumptions appear to violate our genericity hypothesis. However, this is not the case since any generic instance can be transformed, by a linear change of coordinates, into an instance of (2.6)–(2.8) that satisfies (3.2).

Our approach is based on two linear spaces of symmetric matrices,

$$\begin{aligned} \mathcal{U} &= \langle C, A_1, A_2, \dots, A_m \rangle \subset \mathcal{S}^n, \\ \mathcal{V} &= \langle A_2, \dots, A_m \rangle \subset \mathcal{U}. \end{aligned}$$

Thus,  $\mathcal{U}$  is a generic linear subspace of dimension  $m+1$  in  $\mathcal{S}^n$ , and  $\mathcal{V}$  is a generic subspace of codimension 2 in  $\mathcal{U}$ . This specifies a dual pair of flags:

$$\mathcal{V} \subset \mathcal{U} \subset \mathcal{S}^n \quad \text{and} \quad \mathcal{U}^\perp \subset \mathcal{V}^\perp \subset (\mathcal{S}^n)^*. \quad (3.3)$$

In the definition of  $\mathcal{U}$  and  $\mathcal{V}$ , note that  $C$  is included to define  $\mathcal{U}$  and  $A_1$  is excluded to define  $\mathcal{V}$ , this is because we want to discuss the problem in the projective spaces  $\mathbb{P}\mathcal{S}^n$  and  $\mathbb{P}\mathcal{U}$  whose elements are invariant under scaling. So we ignored the constraint  $A_1 \bullet X = 1$ . With these flags the optimality condition (2.6)–(2.7) can be rewritten as

$$X \in \mathcal{V}^\perp \quad \text{and} \quad Y \in \mathcal{U} \quad \text{and} \quad X \cdot Y = 0. \quad (3.4)$$

Our objective is to count the solutions of this system of polynomial equations. Notice that if  $(X, Y)$  is a solution and  $\lambda, \mu$  are non-zero scalars then the pair  $(\lambda X, \mu Y)$  is also a solution. What we are counting are equivalence classes of solutions (3.4) where  $(X, Y)$  and  $(\lambda X, \mu Y)$  are regarded as the same point.

Indeed, from now on, for the rest of this paper, we pass to the usual setting of algebraic geometry: we complexify each of our linear spaces and we consider

$$\mathbb{P}\mathcal{V} \subset \mathbb{P}\mathcal{U} \subset \mathbb{P}\mathcal{S}^n \quad \text{and} \quad \mathbb{P}\mathcal{U}^\perp \subset \mathbb{P}\mathcal{V}^\perp \subset \mathbb{P}(\mathcal{S}^n)^*. \quad (3.5)$$

Each of these six spaces is a **complex projective space**. Note that

$$\dim(\mathbb{P}\mathcal{U}) = m \quad \text{and} \quad \dim(\mathbb{P}\mathcal{V}^\perp) = \binom{n+1}{2} - m. \quad (3.6)$$

If  $\mathcal{W}$  is any of the six linear spaces in (3.3) then we write  $D_{\mathcal{W}}^r$  for the *determinantal variety* of all matrices of rank  $\leq r$  in the corresponding projective space  $\mathbb{P}\mathcal{W}$  in (3.5). Assuming that  $D_{\mathcal{W}}^r$  is non-empty, the codimension of this variety is independent of the choice of the ambient space  $\mathbb{P}\mathcal{W}$ . By [10], we have

$$\text{codim}(D_{\mathcal{W}}^r) = \binom{n-r+1}{2}. \quad (3.7)$$

We write  $\{XY = 0\}$  for the subvariety of the product of projective spaces  $\mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n$  which consists of all pairs  $(X, Y)$  of symmetric  $n \times n$ -matrices whose matrix product is zero. If we fix the ranks of the matrices  $X$  and  $Y$  to be at most  $n-r$  and  $r$  respectively, then we obtain a subvariety

$$\{XY = 0\}^r := \{XY = 0\} \cap (D_{(\mathcal{S}^n)^*}^{n-r} \times D_{\mathcal{S}^n}^r). \quad (3.8)$$

**Lemma 6** *The subvariety  $\{XY = 0\}^r$  is irreducible.*

*Proof.* Our argument is borrowed from Kempf [14]. For a fixed pair of dual bases in  $\mathbb{C}^n$  and  $(\mathbb{C}^n)^*$ , the symmetric matrices  $X$  and  $Y$  define linear maps

$$X : \mathbb{C}^n \rightarrow (\mathbb{C}^n)^* \quad \text{and} \quad Y : (\mathbb{C}^n)^* \rightarrow \mathbb{C}^n.$$

Symmetry implies that  $\ker(X) = \text{Im}(X)^\perp$  and  $\ker(Y) = \text{Im}(Y)^\perp$ . Therefore,  $XY = 0$  holds if and only if  $\ker(X) \supseteq \text{Im}(Y)$ , i.e.  $\ker(Y) \supseteq \ker(X)^\perp$ . For fixed rank  $r$  and a fixed subspace  $K \subset \mathbb{C}^n$ , the set of pairs  $(X, Y)$  such that  $\ker X \supseteq K$  and  $\ker(Y) \supseteq K^\perp$  forms a product of projective linear subspaces of dimension  $\binom{n-r+1}{2} - 1$  and  $\binom{r+1}{2} - 1$  respectively. Over the Grassmannian  $G(r, n)$  of dimension  $r$  subspaces in  $\mathbb{C}^n$ , these triples  $(X, Y, K) \in \mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n \times G(r, n)$  form a fiber bundle that is an irreducible variety. The variety  $\{XY = 0\}^r$  is the image under the projection of that variety onto the first two factors. It is therefore irreducible.  $\square$

Thus, purely set-theoretically, our variety has the irreducible decomposition

$$\{XY = 0\} = \bigcup_{r=1}^{n-1} \{XY = 0\}^r. \quad (3.9)$$

Solving the polynomial equations (3.4) means intersecting these subvarieties of  $\mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n$  with the product of subspaces  $\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}$ . In other words, what is specified by the optimality conditions (2.6)–(2.7) is a point in the variety

$$\begin{aligned} & \{XY = 0\}^r \cap (\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}) \\ &= \{XY = 0\} \cap (D_{\mathcal{V}^\perp}^{n-r} \times D_{\mathcal{U}}^r) \end{aligned} \quad (3.10)$$

which satisfies (2.8). Here  $r$  is the rank of the optimal matrix  $A(\hat{y})$ .

In the special case of linear programming, the matrices  $X$  and  $Y$  in (3.10) are diagonal, and hence  $\{XY = 0\}^r$  is a finite union of linear spaces. Therefore the optimal pair  $(\hat{X}, \hat{Y})$  satisfies some system of linear equations, in accordance with the fact that the algebraic degree of linear programming is equal to one.

The following is our main result in this section.

**Theorem 7.** *For generic  $\mathcal{U}$  and  $\mathcal{V}$ , the variety (3.10) is empty unless Pataki's inequalities (3.1) hold. In that case the variety (3.10) is reduced, non-empty, zero-dimensional, and at each point the rank of  $X$  and  $Y$  is  $n - r$  and  $r$  respectively. The cardinality of this variety depends only on  $m, n$  and  $r$ .*

The rank condition in Theorem 7 is equivalent to the strict complementarity condition stated after Theorem 3. Theorem 7 therefore implies:

**Corollary 8** *The strict complementarity condition holds generically.*

We can now also give an easy proof of Pataki's inequalities:

*Proof of Proposition 5* The inequalities (3.1) are implied by the first sentence of Theorem 7, since the variety (3.10) is not empty.  $\square$

*Proof of Theorem 7* Using (3.6) and (3.7), we can rewrite (3.1) as follows:

$$\text{codim}(D_{S^n}^r) \leq \dim(\mathbb{P}\mathcal{U}) \quad \text{and} \quad \text{codim}(D_{(S^n)^*}^{n-r}) \leq \dim(\mathbb{P}\mathcal{V}^\perp).$$

Pataki's inequalities are obviously necessary for the intersection (3.10) to be non-empty. Suppose now that these inequalities are satisfied.

We claim that the dimension of the variety  $\{XY = 0\}^r$  equals  $\binom{n+1}{2} - 2$ . In particular, this dimension is independent of  $r$ . To verify this claim, we first note that there are  $\binom{n+1}{2} - 1 - \binom{n-r+1}{2}$  degrees of freedom in choosing the matrix  $Y \in D_{S^n}^r$ . For any fixed matrix  $Y$  of rank  $r$ , consider the linear system  $X \cdot Y = 0$  of equations for the entries of  $X$ . Replacing  $Y$  by a diagonal matrix of rank  $r$ , we see that the solution space of this linear system has dimension  $\binom{n-r+1}{2} - 1$ . The sum of these dimensions equals  $\binom{n+1}{2} - 2$  as required. Hence

$$\dim(\{XY = 0\}^r) + \dim(\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}) = \dim(\mathbb{P}(S^n)^* \times \mathbb{P}S^n).$$

By Bertini's theorem [9, Theorem 17.16] for generic choices of the linear spaces  $\mathcal{U}$  and  $\mathcal{V}$ , the intersection of  $\{XY = 0\}^r$  with  $\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}$  is transversal, i.e., in this case finite, each point of intersection is smooth on both varieties, and the tangent spaces intersect transversally. Thus the rank conditions are satisfied at any intersection point. Furthermore, when the intersection is transversal, the number of intersection points is independent of the choice of  $\mathcal{U}$  and  $\mathcal{V}$ .

We therefore exhibit a non-empty transversal intersection. Fix any smooth point  $(X_0, Y_0) \in D_{(S^n)^*}^{n-r} \times D_{S^n}^r$  with  $X_0 \cdot Y_0 = 0$ . This means that  $X_0$  has rank  $n - r$  and  $Y_0$  has rank  $r$ . After a change of bases, we may assume that  $X_0$  is a diagonal matrix with  $r$  zeros and  $n - r$  ones while  $Y_0$  has  $r$  ones and  $n - r$  zeros.

For a generic choice of a subspace  $\mathcal{V}^\perp$  containing  $X_0$  and a subspace  $\mathcal{U}$  containing  $Y_0$ , the intersection  $\{XY = 0\}^r \cap (\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U})$  is transversal away from the point  $(X_0, Y_0)$ . We must show that the intersection is transversal also at  $(X_0, Y_0)$ . We describe the affine tangent

space to  $\{XY = 0\}^r$  at  $(X_0, Y_0)$  in an affine neighborhood of  $(X_0, Y_0) \in \mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n$ . The affine neighborhood is the direct sum of the affine spaces parameterized by

$$\begin{bmatrix} X_{1,1} & \cdots & X_{1,r} & \cdots & \cdots & X_{1,n} \\ \vdots & & & & \ddots & \vdots \\ X_{1,r} & \cdots & X_{r,r} & X_{r,r+1} & \cdots & X_{r,n} \\ X_{1,r+1} & \cdots & X_{r,r+1} & 1 + X_{r+1,r+1} & \cdots & X_{r+1,n} \\ \vdots & & & & \ddots & \vdots \\ X_{1,n-1} & \cdots & \cdots & \cdots & 1 + X_{n-1,n-1} & X_{n-1,n} \\ X_{1,n} & \cdots & \cdots & \cdots & X_{n-1,n} & 1 \end{bmatrix} \quad \text{and} \quad (3.11)$$

$$\begin{bmatrix} 1 & Y_{1,2} & \cdots & \cdots & \cdots & Y_{1,n} \\ Y_{1,2} & 1 + Y_{2,2} & \cdots & \cdots & \cdots & Y_{2,n} \\ \vdots & & & & \ddots & \vdots \\ Y_{1,r} & \cdots & 1 + Y_{r,r} & Y_{r,r+1} & \cdots & Y_{r,n} \\ Y_{1,r+1} & \cdots & Y_{r,r+1} & Y_{r+1,r+1} & \cdots & Y_{r+1,n} \\ \vdots & & & & \ddots & \vdots \\ Y_{1,n} & \cdots & \cdots & Y_{r+1,n} & \cdots & Y_{n,n} \end{bmatrix}. \quad (3.12)$$

In the coordinates of the matrix equation  $XY = 0$ , the linear terms are

$$X_{i,j} \text{ for } i \leq j \leq r, \quad Y_{i,j} \text{ for } r+1 \leq i \leq j \text{ and } X_{i,j} + Y_{i,j} \text{ for } i \leq r < j \leq n$$

These linear forms are independent, and their number equals the codimension of  $\{XY = 0\}^r$  in  $\mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n$ . Hence their vanishing defines the affine tangent space at the smooth point  $(X_0, Y_0)$ . For a generic choice these linear terms are independent also of  $\mathcal{V}$  and  $\mathcal{U}^\perp$ , which assures the transversality at  $(X_0, Y_0)$ .  $\square$

From Theorem 7 we know that the cardinality of the finite variety (3.10) is independent of the choice of generic  $\mathcal{U}$  and  $\mathcal{V}$ , and it is positive if and only if Pataki's inequalities (3.1) hold. We denote this cardinality by  $\delta(m, n, r)$ . Our discussion shows that the number  $\delta(m, n, r)$  coincides with the *algebraic degree of SDP*, which was defined (at the end of Section 2) as the highest degree of the minimal polynomials of the optimal solution coordinates  $\hat{y}_i$  and  $\hat{x}_{jk}$ .

#### 4. A table and some formulas

The general problem of semidefinite programming can be formulated in the following primal-dual form which was derived in Section 3. An instance of SDP is specified by a flag of linear subspaces  $\mathcal{V} \subset \mathcal{U} \subset \mathcal{S}^n$  where  $\dim(\mathcal{V}) = m - 1$  and  $\dim(\mathcal{U}) = m + 1$ , and the goal is to find matrices  $X, Y \in \mathcal{S}^n$  such that

$$X \in \mathcal{V}^\perp \quad \text{and} \quad Y \in \mathcal{U} \quad \text{and} \quad X \cdot Y = 0 \quad \text{and} \quad X, Y \succeq 0. \quad (4.1)$$

Ignoring the inequality constraints  $X, Y \succeq 0$  and fixing the rank of  $X$  to be  $n - r$ , the task amounts to computing the finite variety (3.10). The algebraic degree of SDP, denoted  $\delta(m, n, r)$ , is the cardinality of this projective variety over  $\mathbb{C}$ . A first result about algebraic degree is the following duality relation.

**Proposition 9** *The algebraic degree of SDP satisfies the duality relation*

$$\delta(m, n, r) = \delta\left(\binom{n+1}{2} - m, n, n - r\right). \quad (4.2)$$

	$n = 2$		$n = 3$		$n = 4$		$n = 5$		$n = 6$	
$m$	$r$	degree								
1	1	2	2	3	3	4	4	5	5	6
2	1	2	2	6	3	12	4	20	5	30
3			2	4	3	16	4	40	5	80
			1	4	2	10	3	20	4	35
4					3	8	4	40	5	120
			1	6	2	30	3	90	4	210
5					1	3	2	42	4	16
							3	207	5	96
6							2	30	3	290
							1	8	2	35
7							2	10	3	260
							1	16	2	140
8									3	140
							1	12	2	260
9									1	4
									2	290
									3	3812

**Table 2.** The algebraic degree  $\delta(m, n, r)$  of semidefinite programming

*Proof.* For generic  $C, A_i, b$ , by Corollary 8, the strict complementarity condition holds. So in (2.6)-(2.8), if  $\text{rank}(X^*) = r$ , then  $\text{rank}(A(y^*)) = n - r$ . But the dual problem (2.4)-(2.5) can be written equivalently as some particular primal SDP (2.1)-(2.3) with  $m' = \binom{n+1}{2} - m$  constraints. The roles of  $X$  and  $A(y)$  are reversed. Therefore the duality relation stated in (4.2) holds.  $\square$

A census of all values of the algebraic degree of semidefinite programming for  $m \leq 9$  and  $n \leq 6$  is given in Table 2. Later, we shall propose a formula for arbitrary  $m$  and  $n$ . First, let us explain how Table 2 can be constructed.

Consider the polynomial ring  $\mathbb{Q}[X, Y]$  in the  $n(n + 1)$  unknowns  $x_{ij}$  and  $y_{ij}$ , and let  $\langle XY \rangle$  be the ideal generated by the entries of the matrix product  $XY$ . The quotient  $R = \mathbb{Q}[X, Y]/\langle XY \rangle$  is the homogeneous coordinate ring of the variety  $\{XY = 0\}$ . For fixed rank  $r$  we also consider the prime ideal  $\langle XY \rangle^{\{r\}}$  and the coordinate ring  $R^{\{r\}} = \mathbb{Q}[X, Y]/\langle XY \rangle^{\{r\}}$  of the irreducible component  $\{XY = 0\}^r$  in (3.9). The rings  $R$  and  $R^{\{r\}}$  are naturally graded by the group  $\mathbb{Z}^2$ . The degrees of the generators are  $\deg(x_{ij}) = (1, 0)$  and  $\deg(y_{ij}) = (0, 1)$ .

A convenient tool for computing and understanding the columns of Table 2 is the notion of the *multidegree* in combinatorial commutative algebra [17]. Recall from [17, Section 8.5] that the multidegree of a  $\mathbb{Z}^d$ -graded affine algebra is a homogeneous polynomial in  $d$  unknowns. Its total degree is the height of the multihomogeneous presentation ideal. If  $d = 2$  then we use the term *bidegree* for the multidegree. Let  $\mathcal{C}(R; s, t)$  and  $\mathcal{C}(R^{\{r\}}; s, t)$  be the *bidegree* of the  $\mathbb{Z}^2$ -graded rings  $R$  and  $R^{\{r\}}$  respectively. Since the decomposition (3.9) is equidimensional, additivity of the multidegree [17, Theorem 8.53] implies

$$\mathcal{C}(R; s, t) = \sum_{r=0}^n \mathcal{C}(R^{\{r\}}; s, t).$$

The following result establishes the connection to semidefinite programming:

**Theorem 10.** *The bidegree of the variety  $\{XY = 0\}^r$  is the generating function for the algebraic degree of semidefinite programming. More precisely,*

$$\mathcal{C}(R^{\{r\}}; s, t) = \sum_{m=0}^{\binom{n+1}{2}} \delta(m, n, r) \cdot s^{\binom{n+1}{2}-m} \cdot t^m. \quad (4.3)$$

*Proof.* Fix the ideal  $I = \langle XY \rangle^{\{r\}}$ , and let  $\text{in}(I)$  be its initial monomial ideal with respect to some term order. The bidegree  $\mathcal{C}(R^{\{r\}}; s, t)$  remains unchanged if we replace  $I$  by  $\text{in}(I)$  in the definition of  $R^{\{r\}}$ . The same holds for the right hand side of (4.3) because we can also define  $\delta(m, n, r)$  by using the initial equations  $\text{in}(I)$  in the place of  $\{XY = 0\}^{\{r\}}$  in (3.10). By additivity of the multidegree, it suffices to consider minimal prime ideals of  $\text{in}(I)$ . These are generated by subsets of the unknowns  $x_{ij}$  and  $y_{ij}$ . If such a prime contains  $\binom{n+1}{2} - m$  unknowns  $x_{ij}$  and  $m$  unknowns  $y_{ij}$  then its variety intersects  $\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}$  only if  $\dim(\mathbb{P}\mathcal{U}) = m$ . In this case the intersection consists of one point. Hence the contribution to the bidegree equals  $s^{\binom{n+1}{2}-m} \cdot t^m$  as claimed.  $\square$

This formula (4.3) is useful for practical computations because, in light of the degeneration property [17, Corollary 8.47], the bidegree can be read off from any Gröbner basis for the ideal  $\langle XY \rangle^{\{r\}}$ . Such a Gröbner basis can be computed for small values of  $n$  but no general combinatorial construction of a Gröbner basis is known. Note that if we set  $s = t = 1$  in  $\mathcal{C}(R^{\{r\}}; s, t)$  then we recover the ordinary  $\mathbb{Z}$ -graded degree of the ideal  $\langle XY \rangle^{\{r\}}$ .

*Example 1.* We examine our varieties  $\{XY = 0\}$  for  $n = 4$  in Macaulay 2:

```
R = QQ[x11,x12,x13,x14,x22,x23,x24,x33,x34,x44,
      y11,y12,y13,y14,y22,y23,y24,y33,y34,y44];
X = matrix {{x11, x12, x13, x14},
             {x12, x22, x23, x24},
             {x13, x23, x33, x34},
             {x14, x24, x34, x44}};
Y = matrix {{y11, y12, y13, y14},
             {y12, y22, y23, y24},
             {y13, y23, y33, y34},
             {y14, y24, y34, y44}};
minors(1,X*Y) + minors(2,X) + minors(4,Y); codim oo, degree oo
minors(1,X*Y) + minors(3,X) + minors(3,Y); codim oo, degree oo
minors(1,X*Y) + minors(4,X) + minors(2,Y); codim oo, degree oo
```

This Macaulay 2 code verifies that three of the five irreducible components  $\{XY = 0\}^{\{r\}}$  in (3.9) all have codimension 10, and it computes their  $\mathbb{Z}$ -graded degree:

$$\begin{aligned} \mathcal{C}(R^{\{3\}}; 1, 1) &= \sum_{m=1}^4 \delta(m, 4, 3) &= 4 + 12 + 16 + 8 &= 40, \\ \mathcal{C}(R^{\{2\}}; 1, 1) &= \sum_{m=3}^7 \delta(m, 4, 2) &= 10 + 30 + 42 + 30 + 10 &= 122. \end{aligned}$$

The summands are the entries in the  $n = 4$  column in Table 2. We compute them by modifying the `degree` command so that it outputs the bidegree.  $\square$

We now come to our main result, which is a collection of explicit formulas for many of the entries in Table 2, organized by rows rather than columns.

**Theorem 11.** *The algebraic degree of semidefinite programming,  $\delta(m, n, r)$ , is determined by the following formulas for various special values of  $m, n, r$ :*

1. *If the optimal rank  $r$  equals  $n - 1$  then we have*

$$\delta(m, n, n - 1) = 2^{m-1} \binom{n}{m}.$$

2. If the optimal rank  $r$  equals  $n - 2$  and  $3 \leq m \leq 5$  then we have

$$\begin{aligned}\delta(3, n, n - 2) &= \binom{n+1}{3}, \\ \delta(4, n, n - 2) &= 6\binom{n+1}{4}, \\ \delta(5, n, n - 2) &= 27\binom{n+1}{5} + 3\binom{n+1}{4}.\end{aligned}$$

3. If the optimal rank  $r$  equals  $n - 3$  and  $6 \leq m \leq 9$  then we have

$$\begin{aligned}\delta(6, n, n - 3) &= 2\binom{n+2}{6} + \binom{n+2}{5}, \\ \delta(7, n, n - 3) &= 28\binom{n+3}{7} - 12\binom{n+2}{6}, \\ \delta(8, n, n - 3) &= 248\binom{n+4}{8} - 320\binom{n+3}{7} + 84\binom{n+2}{6}, \\ \delta(9, n, n - 3) &= 1794\binom{n+5}{9} - 3778\binom{n+4}{8} + 2436\binom{n+3}{7} - 448\binom{n+2}{6}.\end{aligned}$$

Theorem 11 combined with Proposition 9 explains all data in Table 2, except for four special values which were first found using computer algebra:

$$\delta(6, 6, 4) = 1400, \quad \delta(7, 6, 4) = 2040, \quad \delta(8, 6, 4) = 2100, \quad \delta(9, 6, 4) = 1470. \quad (4.4)$$

An independent verification of these numbers will be presented in Example 20. This will illustrate our general formula which is conjectured to hold arbitrary values of  $m, n$  and  $r$ . That formula is stated in Theorem 19 and Conjecture 21.

We note that an explicit and completely general formula for the algebraic degree  $\delta(m, n, r)$  was recently found by von Bothmer and Ranestad [25].

## 5. Determinantal varieties and their projective duals

In Theorem 10, the algebraic degree of SDP was expressed in terms of the irreducible components defined by the symmetric matrix equation  $XY = 0$ . In this section we relate this equation to the notion of projective duality, and we interpret  $\delta(m, n, r)$  as the degree of the hypersurface dual to the variety  $D_{\mathcal{U}}^r$ .

Every (complex) projective space  $\mathbb{P}$  has an associated dual projective space  $\mathbb{P}^*$  whose points  $w$  correspond to hyperplanes  $\{w = 0\}$  in  $\mathbb{P}$ , and vice versa. Given any (irreducible) variety  $V \subset \mathbb{P}$ , one defines the *conormal variety*  $CV$  of  $V$  to be the (Zariski) closure in  $\mathbb{P}^* \times \mathbb{P}$  of the set of pairs  $(w, v)$  where  $\{w = 0\} \subset \mathbb{P}$  is a hyperplane tangent to  $V$  at a smooth point  $v \in V$ . The projection of  $CV$  in  $\mathbb{P}^*$  is the *dual variety*  $V^*$  to  $V$ . The fundamental *Biduality Theorem* states that  $CV$  is also the conormal variety of  $V^*$ , and therefore  $V$  is the dual variety to  $V^*$ . For proofs and details see [7, §I.1.3] and [22, §1.3].

In our SDP setting, we write  $\mathbb{P}(\mathcal{S}^n)^*$  for the projective space dual to  $\mathbb{P}\mathcal{S}^n$ . The conormal variety of the determinantal variety  $D_{\mathcal{S}^n}^r$  is well understood:

**Proposition 12** [7, Proposition I.4.11] *The irreducible variety  $\{XY = 0\}^r$  in  $\mathbb{P}(\mathcal{S}^n)^* \times \mathbb{P}\mathcal{S}^n$  coincides with the conormal variety of the determinantal variety  $D_{\mathcal{S}^n}^r$  and likewise of  $D_{(\mathcal{S}^n)^*}^{n-r}$ . In particular  $D_{(\mathcal{S}^n)^*}^{n-r}$  is the dual variety to  $D_{\mathcal{S}^n}^r$ .*

*Proof.* Consider a symmetric  $n \times n$ -matrices  $Y$  of rank  $r$ . We may assume that  $Y$  is diagonal, with the first  $r$  entries in the diagonal equal to 1, and the remaining equal to 0. In the affine neighborhood of  $Y$ , given by  $Y_{1,1} \neq 0$ , the matrices have the form (3.12). The determinantal subvariety  $D_{\mathcal{S}^n}^r$  intersects this neighborhood in the locus where the size  $r+1$  minors vanish. The linear parts of these minors specify the matrices  $X$  that define the hyperplanes tangent to  $D_{\mathcal{S}^n}^r$  at  $Y$ . The only such minors with a linear part are those that contain the upper left  $r \times r$  submatrix. Furthermore, their linear part is generated by the entries of the lower right  $(n-r) \times (n-r)$  matrix. But the matrices whose only nonzero entries are in this lower right submatrix are precisely those that satisfy our matrix equation  $X \cdot Y = 0$ .  $\square$

**Theorem 13.** *Let  $\mathcal{U}$  be a generic  $(m+1)$ -dimensional linear subspace of  $\mathcal{S}^n$ , and consider the determinantal variety  $D_{\mathcal{U}}^r$  of symmetric matrices of rank at most  $r$  in  $\mathbb{P}\mathcal{U}$ . Then its dual variety  $(D_{\mathcal{U}}^r)^*$  is a hypersurface if and only if Pataki's inequalities (3.1) hold, and, in this case, the algebraic degree of semidefinite programming coincides with the degree of the dual hypersurface:*

$$\delta(m, n, r) = \deg(D_{\mathcal{U}}^r)^*.$$

*Proof.* Recall the codimension 2 inclusions  $\mathcal{V} \subset \mathcal{U}$  in  $\mathcal{S}^n$  and  $\mathcal{U}^\perp \subset \mathcal{V}^\perp$  in  $\mathcal{S}^{n*}$ . The space of linear forms  $\mathcal{U}^*$  is naturally identified with the quotient space  $\mathcal{S}^{n*}/\mathcal{U}^\perp$ , and hence  $\mathbb{P}(\mathcal{S}^{n*}/\mathcal{U}^\perp) = \mathbb{P}\mathcal{U}^*$ . The image of the induced rational map  $\mathbb{P}\mathcal{V}^\perp \rightarrow \mathbb{P}\mathcal{U}^*$  is the projective line  $\mathbb{P}^1 := \mathbb{P}(\mathcal{V}^\perp/\mathcal{U}^\perp)$ . The points on the line  $\mathbb{P}^1$  correspond to the hyperplanes in  $\mathbb{P}\mathcal{U}$  that contain the codimension 2 subspace  $\mathbb{P}\mathcal{V}$ . The map  $\mathbb{P}\mathcal{V}^\perp \rightarrow \mathbb{P}\mathcal{U}^*$  induces a map in the first factor

$$\pi : \mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U} \rightarrow \mathbb{P}^1 \times \mathbb{P}\mathcal{U} \subset \mathbb{P}\mathcal{U}^* \times \mathbb{P}\mathcal{U}.$$

Note that the rational map  $\pi$  is only defined outside  $\mathbb{P}\mathcal{U}^\perp \times \mathbb{P}\mathcal{U}$ .

We already know that  $\delta(m, n, r)$  is the cardinality of the finite variety

$$Z := \{XY = 0\}^r \cap (\mathbb{P}\mathcal{V}^\perp \times \mathbb{P}\mathcal{U}) = \{XY = 0\} \cap (D_{\mathcal{V}^\perp}^{n-r} \times D_{\mathcal{U}}^r).$$

Since  $\mathcal{U}^\perp$  is generic inside  $\mathcal{V}^\perp$ , none of the points of  $Z$  lies in  $\mathbb{P}\mathcal{U}^\perp \times \mathbb{P}\mathcal{U}$ , so we can apply  $\pi$  to  $Z$ , and the image  $\pi(Z)$  is a finite subset of  $\mathbb{P}^1 \times \mathbb{P}\mathcal{U} \subset \mathbb{P}\mathcal{U}^* \times \mathbb{P}\mathcal{U}$ .

We next prove that  $Z$  and  $\pi(Z)$  have the same cardinality. By Proposition 12, a point on  $\{XY = 0\}^r$  is a pair  $(X, Y)$  where the hyperplane  $\{X = 0\}$  is tangent to the determinantal variety  $D_{\mathcal{S}^n}^r$  at the point  $Y$ . Thus, if  $(X_0, Y_0)$  and  $(X_1, Y_0)$  are distinct points in  $Z$  that have the same image under  $\pi$ , then both  $\{X_0 = 0\}$  and  $\{X_1 = 0\}$  contain the tangent space to  $D_{\mathcal{S}^n}^r$  at  $Y_0$ . The same holds for  $\{sX_0 + tX_1 = 0\}$  for any  $s, t \in \mathbb{C}$ . Hence  $Z$  contains the entire line  $\{(sX_0 + tX_1, Y_0) : s, t \in \mathbb{C}\}$ , which is a contradiction to  $Z$  being finite. Therefore  $\pi$  restricted to  $Z$  is one to one, and we conclude  $\#\pi(Z) = \delta(m, n, r)$ .

A point  $(X, Y)$  in  $Z$  represents a hyperplane  $\{X = 0\}$  in  $\mathbb{P}\mathcal{S}^n$  that contains the subspace  $\mathbb{P}\mathcal{V}$  and is tangent to  $D_{\mathcal{S}^n}^r$  at a point  $Y$  in  $\mathbb{P}\mathcal{U}$ , i.e. it is tangent to the determinantal variety  $D_{\mathcal{U}}^r$  at this point. Consider the map  $\{X = 0\} \mapsto \{X = 0\} \cap \mathbb{P}\mathcal{U}$  which takes hyperplanes in  $\mathbb{P}\mathcal{S}^n$  to hyperplanes in  $\mathbb{P}\mathcal{U}$ . This map is precisely the rational map  $\mathbb{P}\mathcal{V}^\perp \rightarrow \mathbb{P}\mathcal{U}^*$  defined above. The image of  $(X, Y) \in Z$  under  $\pi$  thus represents a hyperplane  $\{X = 0\} \cap \mathbb{P}\mathcal{U}$  that contains the codimension 2 subspace  $\mathbb{P}\mathcal{V}$  and is tangent to  $D_{\mathcal{U}}^r$  at a point  $Y$ . The hyperplanes in  $\mathbb{P}\mathcal{U}$  that contains  $\mathbb{P}\mathcal{V}$  form the projective line  $\mathbb{P}^1 := \mathbb{P}(\mathcal{V}^\perp/\mathcal{U}^\perp)$  in  $\mathbb{P}\mathcal{U}^*$ , so  $\pi(Z)$  is simply the set of all hyperplanes in that  $\mathbb{P}^1$  which are tangent to  $D_{\mathcal{U}}^r$ . Equivalently,  $\pi(Z)$  is the intersection of the dual variety  $(D_{\mathcal{U}}^r)^*$  with a general line in  $\mathbb{P}\mathcal{U}^*$ . Hence  $\pi(Z)$  is nonempty if and only if  $(D_{\mathcal{U}}^r)^*$  is a hypersurface, in which case its cardinality  $\delta(m, n, r)$  is the degree of that hypersurface. The first sentence in Theorem 7 says that  $Z \neq \emptyset$  if and only if Pataki's inequalities hold.  $\square$

The examples in the Introduction serve as an illustration for Theorem 13. In Example 1, the variety  $D_{\mathcal{U}}^1$  is a cubic Vinnikov curve and its dual curve  $(D_{\mathcal{U}}^1)^*$  has degree  $\delta(2, 3, 1) = 6$ . Geometrically, a general line meets the (singular) curve in Figure 2 in six points, at least two of which are complex. A pencil of parallel lines in Figure 1 contains six that are tangent to the cubic.

In Example 2, the variety  $D_{\mathcal{U}}^2$  is the Cayley cubic in Figure 3, and the variety  $D_{\mathcal{U}}^1$  consists of its four singular points. The dual surface  $(D_{\mathcal{U}}^2)^*$  is a Steiner surface of degree 4, and the dual surface  $(D_{\mathcal{U}}^1)^*$  consists of four planes in  $\mathbb{P}\mathcal{U}^* \simeq \mathbb{P}^3$ . Thinking of  $D_{\mathcal{U}}^2$  as a limit of smooth cubic surfaces, we see that each of the planes in  $(D_{\mathcal{U}}^1)^*$  should be counted with multiplicity two, as the degree of the surface dual to a smooth cubic surface is  $12 = 4 + 2 \cdot 4$ .

In general, the degree of the hypersurface  $(D_{\mathcal{U}}^r)^*$  depends crucially on the topology and singularities of the primal variety  $D_{\mathcal{U}}^r$ . In what follows, we examine these features of determinantal varieties, starting with the case  $\mathcal{U} = \mathcal{S}^n$ .

**Proposition 14** *The determinantal variety  $D_{\mathcal{S}^n}^r$  of symmetric matrices of rank at most  $r$  has codimension  $\binom{n-r+1}{2}$  and is singular precisely along the subvariety  $D_{\mathcal{S}^n}^{r-1}$  of matrices of rank at most  $r-1$ . The degree of  $D_{\mathcal{S}^n}^r$  equals*

$$\deg(D_{\mathcal{S}^n}^r) = \prod_{j=0}^{n-r-1} \frac{\binom{n+j}{n-r-j}}{\binom{2j+1}{j}}.$$

*Proof.* The codimension, and the facts that  $D_{\mathcal{S}^n}^r$  is singular along  $D_{\mathcal{S}^n}^{r-1}$ , appears in [9, Example 14.16]. The formula for the degree is [10, Proposition 12].  $\square$

When  $D_{\mathcal{U}}^r$  is finite, then the above formula suffices to determine our degree.

**Corollary 15** *The algebraic degree of semidefinite programming satisfies*

$$\delta(m, n, r) = \prod_{j=0}^{n-r-1} \frac{\binom{n+j}{n-r-j}}{\binom{2j+1}{j}} \quad \text{provided } m = \binom{n-r+1}{2}.$$

*Proof.* If  $\mathcal{U} \subset \mathcal{S}^n$  is a generic subspace of dimension  $\binom{n-r+1}{2} + 1$ , then  $\mathbb{P}\mathcal{U}$  and  $D_{\mathcal{S}^n}^r$  have complementary dimension in  $\mathbb{P}\mathcal{S}^n$ , so  $D_{\mathcal{U}}^r = \mathbb{P}\mathcal{U} \cap D_{\mathcal{S}^n}^r$  is finite and reduced, with cardinality equal to the degree of  $D_{\mathcal{S}^n}^r$ . The dual variety of a finite set in  $\mathbb{P}\mathcal{U}$  is a finite union of hyperplanes in  $\mathbb{P}\mathcal{U}^*$ , one for each point.  $\square$

For  $m = n = 3$  and  $r = 1$  this formula gives us  $\delta(3, 3, 1) = 4$ . This is the number of singular points on the Cayley cubic surface in Example 2.

More generally, whenever  $D_{\mathcal{U}}^r$  is smooth (i.e. when  $D_{\mathcal{U}}^{r-1} = \emptyset$ ), the degree of the dual hypersurface may be computed by the following *Class Formula*. For any smooth variety  $X \subset \mathbb{P}^m$  we write  $\chi(X)$  for its *Euler number*, which is its topological Euler-Poincaré characteristic. Likewise, we write  $\chi(X \cap H)$  and  $\chi(X \cap H \cap H')$  for the Euler number of the intersection of  $X$  with one or two general hyperplanes  $H$  and  $H'$  in  $\mathbb{P}^m$  respectively.

**Proposition 16** (Class Formula, [22, Theorem 10.1]) *If  $X$  is any smooth subvariety of  $\mathbb{P}^m$  whose projective dual variety  $X^*$  is a hypersurface, then*

$$\deg X^* = (-1)^{\dim(X)} \cdot (\chi(X) - 2 \cdot \chi(X \cap H) + \chi(X \cap H \cap H'))$$

This formula is best explained in the case when  $X$  is a curve, so that  $X \cap H$  is a finite set of points. The Euler number of  $X \cap H$  is its cardinality, i.e. the degree of  $X$ . Furthermore  $X \cap H \cap H' = \emptyset$ , so the Class Formula reduces to

$$\deg X^* = -\chi(X) + 2 \cdot \deg X. \quad (5.1)$$

To see this, we compute the Euler number  $\chi(X)$  using a general pencil of hyperplanes, namely those containing the codimension 2 subspace  $H \cap H' \subset \mathbb{P}^m$ . Precisely  $\hat{d} = \deg X^*$  of the hyperplanes in this pencil are tangent to  $X$ , and each of these hyperplanes will be tangent at one point and intersect  $X$  in  $\deg X - 2$  further points. So the Euler number is  $\deg X - 1$  for each of these hyperplane sections. The other hyperplane sections all have Euler number  $\deg X$  and are parameterized by the complement of  $\hat{d}$  points in a  $\mathbb{P}^1$ . By the multiplicative property of the Euler number, the union of the smooth hyperplane sections has Euler number  $(\chi(\mathbb{P}^1) - \hat{d}) \cdot (\deg X)$ . By additivity of the Euler number on a disjoint union, we get the Class Formula (5.1) for curves:  $\chi(X) = (\chi(\mathbb{P}^1) - \hat{d}) \cdot (\deg X) + \hat{d} \cdot (\deg X - 1) = 2 \cdot \deg X - \hat{d}$ .

The number  $\chi(D_{\mathcal{U}}^r)$  depends only on  $m, n$  and  $r$ , when  $\mathcal{U}$  is general, so we set  $\chi(m, n, r) := \chi(D_{\mathcal{U}}^r)$ . When  $H$  and  $H'$  are general, the varieties  $D_{\mathcal{U}}^r \cap H$  and  $D_{\mathcal{U}}^r \cap H \cap H'$  are again general determinantal varieties, consisting of matrices of rank  $\leq r$  in a codimension 1 (resp. codimension 2) subspace of  $\mathcal{U}$ . The Class Formula therefore implies the following result.

**Corollary 17** Suppose that  $\binom{n-r+1}{2} \leq m < \binom{n-r+2}{2}$ . Then we have

$$\delta(m, n, r) = (-1)^{m-\binom{n-r+1}{2}} \cdot (\chi(m, n, r) - 2 \cdot \chi(m-1, n, r) + \chi(m-2, n, r)).$$

*Proof.* The determinantal variety  $D_{\mathcal{U}}^r \subset \mathbb{P}\mathcal{U} = \mathbb{P}^m$  for a generic  $\mathcal{U} \subset \mathcal{S}^n$  is nonempty if and only if  $m \geq \text{codim } D_{\mathcal{U}}^r = \binom{n-r+1}{2}$ , and it is smooth as soon as  $D_{\mathcal{U}}^{r-1}$  is empty, i.e. when  $m < \binom{n-r+2}{2}$ . Therefore the Class Formula applies and gives the expression for the degree of the dual hypersurface  $(D_{\mathcal{U}}^r)^*$ .  $\square$

The duality in Proposition 9 states  $\delta(m, n, r) = \delta(\binom{n+1}{2} - m, n, n-r)$ . So, as long as one of the two satisfies the inequalities of the Corollary 17, the Class Formula computes the dual degree. In Table 2 this covers all cases except  $\delta(6, 6, 4)$ , and it covers all the cases of Theorem 11. So, for the proof of Theorem 11, it remains to compute the Euler number for a smooth  $D_{\mathcal{U}}^r$ . We close with the remark that the Class Formula fails when  $D_{\mathcal{U}}^r$  is singular.

## 6. Proofs and a conjecture

The proof of Theorem 11 has been reduced, by Corollary 17, to computing the Euler number for a smooth degeneracy locus of symmetric matrices. We begin by explaining the idea of the proof in the case of the first formula.

*Proof of Theorem 11 (1)* Since  $\delta(m, n, n-1) = \delta(\binom{n+1}{2} - m, n, 1)$ , by Proposition 9, we may consider the variety of symmetric rank 1 matrices, which is smooth and coincides with the second Veronese embedding of  $\mathbb{P}^{n-1}$ . The determinantal variety  $D_{\mathcal{U}}^1$  is thus a linear section of this Veronese embedding of  $\mathbb{P}^{n-1}$ . Equivalently,  $D_{\mathcal{U}}^1$  is the Veronese image of a complete intersection of  $m-1$  quadrics in  $\mathbb{P}^{n-1}$ . Our goal is to compute the Euler number of  $D_{\mathcal{U}}^1$ .

By the Gauss-Bonnet formula, the Euler number of a smooth variety is the degree of the top Chern class of its tangent bundle. Let  $h$  be the class of a hyperplane in  $\mathbb{P}^{n-1}$ . The tangent bundle of  $D_{\mathcal{U}}^1$  is the quotient of the tangent bundle of  $\mathbb{P}^{n-1}$  restricted to  $D_{\mathcal{U}}^1$  and

the normal bundle of  $D_{\mathcal{U}}^1$  in  $\mathbb{P}^{n-1}$ . The total Chern class of our determinantal manifold  $D_{\mathcal{U}}^1$  is therefore the quotient

$$c(D_{\mathcal{U}}^1) = \frac{(1+h)^n}{(1+2h)^{m-1}},$$

and the top Chern class  $c_{n-m}(D_{\mathcal{U}}^1)$  is the degree  $n-m$  term in  $c(D_{\mathcal{U}}^1)$ . Similarly the top Chern class of  $D_{\mathcal{U}}^1 \cap H$  and  $D_{\mathcal{U}}^1 \cap H \cap H'$  is the degree  $n-m$  terms of

$$\frac{(1+h)^n}{(1+2h)^m} \cdot 2h \quad \text{and} \quad \frac{(1+h)^n}{(1+2h)^{m+1}} \cdot 4h^2,$$

where the last factor indicates that we evaluate these classes on  $D_{\mathcal{U}}^1$  and use the fact that  $H = 2h$ . By Proposition 16, the dual degree  $\deg(D_{\mathcal{U}}^1)^*$  is obtained by evaluating  $(-1)^{n-m}$  times the degree  $n-m$  term in the expression

$$\frac{(1+h)^n}{(1+2h)^{m-1}} - 2 \frac{(1+h)^n}{(1+2h)^m} 2h + \frac{(1+h)^n}{(1+2h)^{m+1}} 4h^2 = \frac{(1+h)^n}{(1+2h)^{m+1}}.$$

That term equals  $\binom{n}{m} \cdot h^{n-m}$ . Since the degree of our determinantal variety equals  $\deg D_{\mathcal{U}}^1 = \int_{D_{\mathcal{U}}^1} h^{n-m} = 2^{m-1}$ , we conclude  $\deg(D_{\mathcal{U}}^1)^* = \binom{n}{m} \cdot 2^{n-m}$ .  $\square$

In the general case we rely on a formula of Piotr Pragacz [19]. He uses *Schur Q-functions* to define an intersection number on  $\mathbb{P}^m$  with support on the degeneracy locus of a symmetric morphism of vector bundles. Our symmetric determinantal varieties are special cases of this. Pragacz' intersection number does not depend on the smoothness of the degeneracy locus, but only in the smooth case he shows that the intersection number equals the Euler number. By Corollary 17 we then obtain a formula for  $\delta(m, n, r)$  in the smooth range.

To present Pragacz' formula, we first need to fix our notation for partitions. A *partition*  $\lambda$  is a finite weakly decreasing sequence of nonnegative integers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$ . In contrast to the usual conventions [16], we distinguish between partitions that differ only by a string of zeros in the end. For instance, we consider  $(2, 1)$  and  $(2, 1, 0)$  to be distinct partitions. The *length* of a partition  $\lambda$  is the number  $k$  of its parts, while the *weight*  $|\lambda| = \lambda_1 + \dots + \lambda_k$  is the sum of its parts. The sum of two partitions is the partition obtained by adding the corresponding  $\lambda_i$ . A partition is *strict* if  $\lambda_{i-1} > \lambda_i$  for all  $i = 1, \dots, k-1$ . The special partitions  $(k, k-1, k-2, \dots, 1)$  and  $(k, k-1, k-2, \dots, 1, 0)$  are denoted by  $\rho(k)$  and  $\rho_0(k)$  respectively.

Now, let  $E$  be a rank  $n$  vector bundle on  $\mathbb{P}^m$  with Chern roots  $x_1, \dots, x_n$ , and let  $\phi : E^* \rightarrow E$  be a symmetric morphism. Consider the degeneracy locus  $D^r(\phi)$  of points in  $\mathbb{P}^m$  where the rank of  $\phi$  is at most  $r$ . For any strict partition  $\lambda$  let  $Q_\lambda(E)$  be the Schur *Q*-function in the Chern roots (see [6, 16, 19] for definitions). Thus  $Q_\lambda(E)$  is a symmetric polynomial in  $x_1, \dots, x_n$  of degree equal to the weight of  $\lambda$ . Pragacz defines the intersection number

$$e(D^r(\phi)) = \int_{\mathbb{P}^m} \sum_{\lambda} (-1)^{|\lambda|} \cdot ((\lambda + \rho_0(n-r-1))) \cdot Q_{(\lambda+\rho(n-r))}(E) \cdot c(\mathbb{P}^m), \quad (6.1)$$

where the sum is over all partitions  $\lambda$  of length  $n-r$  and weight  $|\lambda| \leq m - \binom{n-r+1}{2}$ . Let us carefully describe the ingredients of the formula. The factor  $c(\mathbb{P}^m)$  is the total Chern class  $(1+h)^{m+1}$  of  $\mathbb{P}^m$ . For any strict partition  $\lambda := (\lambda_1 > \lambda_2 > \dots > \lambda_k \geq 0)$ , the factor  $((\lambda))$  is an integer which is defined as follows. It depends on  $k$  whether or not  $\lambda_k = 0$ . If the number  $k$  is 1 or 2 then we set  $((i)) := 2^i$  and

$$((i, j)) := \binom{i+j}{i} + \binom{i+j}{i-1} + \dots + \binom{i+j}{j+1}.$$

In general, when  $k$  is even, we set

$$\langle(\lambda_1, \dots, \lambda_k)\rangle := \text{Pfaff}\left(\langle(\lambda_s, \lambda_t)\rangle\right)_{s < t}.$$

Here ‘‘Pfaff’’ denotes the *Pfaffian* (i.e. the square root of the determinant) of a skew-symmetric matrix of even size. Finally, when  $k$  is odd, we set

$$\langle(\lambda_1, \dots, \lambda_k)\rangle := \sum_{j=1}^k (-1)^{j-1} \cdot 2^{\lambda_j} \cdot \langle(\lambda_1, \dots, \widehat{\lambda_j}, \dots, \lambda_k)\rangle.$$

**Proposition 18** (Pragacz, [19, Prop. 7.13]) *If the degeneracy locus  $D^r(\phi)$  is smooth and of maximal codimension  $\binom{n-r+1}{2}$  then  $\chi(D^r(\phi)) = e(D^r(\phi))$ .*

In our situation, the morphism  $\phi$  arises from the space  $\mathcal{U}$  and is given by a symmetric matrix whose entries are linear forms on  $\mathbb{P}^m$ . We thus apply the trick, used in [10] and also in [6, Section 6.4], of formally writing

$$E = \mathcal{O}_{\mathbb{P}^m}(\frac{h}{2}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(\frac{h}{2}) = n \mathcal{O}_{\mathbb{P}^m}(\frac{h}{2}).$$

Here  $\phi$  is a map  $n \mathcal{O}_{\mathbb{P}^m}(-\frac{h}{2}) \rightarrow n \mathcal{O}_{\mathbb{P}^m}(\frac{h}{2})$  and the determinantal variety  $D_{\mathcal{U}}^r$  is the locus of points where this map has rank at most  $r$ . The  $n$  Chern roots of the split vector bundle  $E$  are all equal to  $h/2$ . Thus, in applying Pragacz’ formula (6.1), we restrict the Schur Q-functions to the diagonal  $x_1 = \dots = x_n = h/2$ .

The result of this specialization of the Schur Q-function is an expression

$$Q_{\lambda}(E) = b_{\lambda}(n) \cdot h^{|\lambda|}, \quad (6.2)$$

where  $b_{\lambda}(n)$  is a polynomial in  $n$  with  $b(\mathbb{Z}) \subseteq \mathbb{Z}[1/2]$ . We multiply (6.2) with the complementary Chern class of  $\mathbb{P}^m$ , which is the expression

$$c_{|\lambda|}(\mathbb{P}^m) = \binom{m+1}{m - \binom{n-r+1}{2} - |\lambda|} \cdot h^{m-|\lambda|}$$

Pragacz’ intersection number (6.1) now evaluates to

$$\sum_{\lambda} (-1)^{|\lambda|} \cdot \langle(\lambda + \rho_0(n-r-1))\rangle \cdot b_{(\lambda+\rho(n-r))}(n) \cdot \binom{m+1}{m - \binom{n-r+1}{2} - |\lambda|}. \quad (6.3)$$

We abbreviate the expression (6.3) by  $e(m, n, r)$ . Note that  $e(m, n, r)$  is a polynomial in  $n$  for fixed  $m$  and  $r$ . In the smooth range we can now apply Corollary 17 to obtain a formula for the degree of the dual variety  $(D_{\mathcal{U}}^r)^*$ :

$$\delta(m, n, r) = (-1)^{(m - \binom{r+1}{2})} \cdot (e(m, n, r) - 2 \cdot e(m-1, n, r) + e(m-2, n, r)).$$

This yields a formula for the algebraic degree of semidefinite programming:

**Theorem 19.** *If  $\binom{n-r+1}{2} \leq m < \binom{n-r+2}{2}$  then*

$$\delta(m, n, r) = (-1)^d \cdot \sum_{\lambda} (-1)^{|\lambda|} \cdot \langle(\lambda + \rho_0(n-r-1))\rangle \cdot b_{(\lambda+\rho(n-r))}(n) \cdot \binom{m-1}{d-|\lambda|},$$

where  $d = m - \binom{n-r+1}{2}$  is the dimension of the variety  $D_{\mathcal{U}}^r$  and the sum is over all partitions  $\lambda$  of length  $n-r$  and weight  $|\lambda| \leq m - \binom{n-r+1}{2}$ .

*Proof.* It remains only to apply Corollary 17. Comparing the formulas for  $e(m, n, r)$ ,  $e(m - 1, n, r)$  and  $e(m - 2, n, r)$ , the difference is in the number of summands and in the binomial coefficient in the last factor. But the relation

$$\binom{m+1}{k} - 2\binom{m}{k-1} + \binom{m-1}{k-2} = \binom{m-1}{k}$$

holds whenever  $k > 1$ , while  $\binom{m+1}{1} - 2\binom{m}{0} = \binom{m-1}{1}$  and  $\binom{m+1}{0} = \binom{m-1}{0}$ . So the formula for  $\delta(m, n, r)$  reduces to the one claimed in the theorem.  $\square$

The formula for  $\delta(m, n, r)$  in Theorem 19 is explicit but quite impractical. To make it more useful, we present a rule for computing the polynomials  $b_{(i_1, \dots, i_k)}(n)$  for any  $i_1 > \dots > i_k \geq 0$ . First, let  $b_i(n)$  be the coefficient of  $h^i$  in

$$\frac{(1+h/2)^n}{(1-h/2)^n} = b_0(n) + b_1(n) \cdot h + \dots + b_k(n) \cdot h^k + \dots$$

The coefficient  $b_i = b_i(n)$  is a polynomial of degree  $i$  in the unknown  $n$ , namely,

$$b_0 = 1, \quad b_1 = n, \quad b_2 = \frac{1}{2}n^2, \quad b_3 = \frac{1}{6}n^3 + \frac{1}{12}n, \quad b_4 = \frac{1}{24}n^4 + \frac{1}{12}n^2, \quad (6.4)$$

$$b_5 = \frac{1}{120}n^5 + \frac{1}{24}n^3 + \frac{1}{80}n, \quad b_6 = \frac{1}{720}n^6 + \frac{1}{72}n^4 + \frac{23}{1440}n^2, \dots \quad (6.5)$$

We next set  $b_{(i,0)}(n) = b_i(n)$  and

$$b_{(i,j)}(n) = b_i(n) \cdot b_j(n) - 2 \cdot \sum_{k=1}^j (-1)^{k-1} \cdot b_{i+k}(n) \cdot b_{j-k}(n).$$

The general formula is now given by distinguishing three cases:

$$\begin{aligned} b_{(i_1, \dots, i_k)}(n) &= \text{Pfaff}(b_{(i_s, i_t)})_{s < t} && \text{if } k \text{ is even,} \\ b_{(i_1, \dots, i_k)}(n) &= b_{(i_1, \dots, i_k, 0)}(n) && \text{if } k \text{ is odd and } i_k > 0, \\ b_{(i_1, \dots, i_{k-1}, 0)}(n) &= b_{(i_1, \dots, i_{k-1})}(n) && \text{if } k \text{ is odd and } i_k = 0. \end{aligned}$$

*Proof of Theorem 11 (2),(3)* All seven formulas are gotten by specializing the formula in Theorem 19. Let us begin with the first one where  $m = 3$  and  $r = n - 2$ . Here  $D_{\mathcal{U}}^{n-2}$  is 0-dimensional and both the Euler number and the dual degree computes the number of points in  $D_{\mathcal{U}}^{n-2}$ . The formula says

$$\delta(3, n, n - 2) = \sum_{\lambda} (-1)^{|\lambda|} \cdot ((\lambda_1 + 1, \lambda_2)) \cdot b_{(\lambda_1 + 2, \lambda_1 + 1)}(n) \cdot \binom{2}{0 - |\lambda|}.$$

The only partition  $(\lambda_1, \lambda_2)$  in the sum is  $(0, 0)$ . Hence  $\delta(3, n, n - 2)$  equals

$$((1, 0)) \cdot b_{(2,1)}(n) = b_2(n)b_1(n) - 2b_3(n) = n \frac{n^2}{2} - 2 \frac{2n^3 + n}{12} = \binom{n+1}{3}$$

Next consider the case  $m = 4$  and  $r = n - 2$ . Here  $D_{\mathcal{U}}^{n-2}$  has dimension  $d = 1$ , and the sum in our formula is over the two partitions  $\lambda = (0, 0)$  and  $\lambda = (1, 0)$ :

$$\begin{aligned} \delta(4, n, n - 2) &= -3 \cdot ((1, 0)) \cdot b_{(2,1)}(n) + ((2, 0)) \cdot b_{(3,1)}(n) \\ &= -3 \cdot (b_2(n)b_1(n) - 2b_3(n)) + 3 \cdot (b_3(n)b_1(n) - 2b_4(n)) \end{aligned}$$

If we substitute (6.4) into this expression, then we obtain  $6\binom{n+1}{4}$  as desired.

The other five cases are similar. We derive only one more: for  $m = 8$  and  $r = n - 3$ , the sum is over four partitions  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(2, 0, 0)$  and  $(1, 1, 0)$ :

$$\begin{aligned}\delta(8, n, n - 3) &= 21 \cdot ((2, 1, 0)) \cdot b_{321}(n) - 7 \cdot ((3, 1, 0)) \cdot b_{421}(n) \\ &\quad + 1 \cdot ((4, 1, 0)) \cdot b_{521}(n) + 1 \cdot ((3, 2, 0)) \cdot b_{431}(n).\end{aligned}$$

After applying the Pfaffian formulas

$$b_{ijk} = b_{ij} \cdot b_{k0} - b_{ik}b_{j0} + b_{jk}b_{i0} \quad \text{and} \quad ((i, j, k)) = 2^i((j, k)) - 2^j((i, k)) + 2^k((i, j)),$$

we substitute (6.4)–(6.5) into this expression and obtain the desired result.  $\square$

**Example 20** Consider the four special entries of Table 2 which are listed in (4.4). Using the duality (4.2) of Proposition 9, we rewrite these values as

$$\delta(m, 6, 2) = \delta(21 - m, 6, 4) \quad \text{for } m = 15, 14, 13, 12.$$

The last three cases satisfy our hypothesis  $10 = \binom{6-2+1}{2} \leq 21 - m < \binom{6-2+2}{2} = 15$ , so Theorem 19 applies and furnishes an independent proof of the correctness of these values in (4.4). The only remaining entry in Table 2 is  $\delta(6, 6, 4) = 1400$ . The formula in Theorem 19 correctly predicts that value, too.  $\square$

We conjecture that the formula of Theorem 19 holds in the general singular case. The formula does indeed make sense without the smoothness assumption, and it does give the correct number in all cases that we have checked. Experts in symmetric function theory might find it an interesting challenge to verify that the conjectured formula actually satisfies the duality relation (4.2).

**Conjecture 21** The formula for the algebraic degree of semidefinite programming in Theorem 19 holds without the restriction in the range of  $m$ .

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## References

1. F. Alizadeh, J. Haeberly and M. Overton: Complementarity and nondegeneracy in semidefinite programming, *Mathematical Programming* **77** (1997) 111–128.
2. C. Bajaj: The algebraic degree of geometric optimization problems. *Discrete Comput. Geom.* **3** (1988) 177–191.
3. F. Catanese, S. Hoşten, A. Khetan and B. Sturmfels: The maximum likelihood degree, *American Journal of Mathematics*, **128** (2006) 671–697.
4. R. Datta: Universality of Nash equilibria, *Math. Oper. Res.* **28** (2003) 424–432.
5. J. De Loera, J. Rambau and F. Santos, *Triangulations: Applications, Structures, and Algorithms*. Algorithms and Computation in Mathematics. Springer, to appear.
6. W. Fulton and P. Pragacz: *Schubert Varieties and Degeneracy loci*, Lecture Notes in Mathematics **1689**. Springer, New York, 1998.
7. I. Gelfand, M. Kapranov and A. Zelevinsky: *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
8. D. Grayson and M. Stillman: Macaulay 2, a software system for research in algebraic geometry, Available at <http://www.math.uiuc.edu/Macaulay2/>.

9. J. Harris: *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics **113**, Springer, New York, 1992.
10. J. Harris and L. Tu: On symmetric and skew-symmetric determinantal varieties, *Topology* **23** (1984) 71–84.
11. W. Helton and V. Vinnikov: Linear matrix inequality representation of sets, *Comm. Pure Appl. Math* **60** (2007) 654–674.
12. S. Hoşten, A. Khetan and B. Sturmfels: Solving the likelihood equations, *Foundations of Computational Mathematics* **5** (2005) 389–407.
13. T. Jęzefiak, A. Lascoux, P. Pragacz: Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981) 662–673.
14. G. Kempf: Images of homogeneous vector bundles and varieties of complexes, *Bulletin of the American Math. Society* **81** (1975) 900–901.
15. A. Lewis, P. Parrilo and M. Ramana: The Lax conjecture is true, *Proceedings of the American Math. Society* **133** (2005) 2495–2499.
16. I.G. Macdonald: *Symmetric functions and Hall polynomials -2nd ed.*, Clarendon Press, Oxford, 1995.
17. E. Miller and B. Sturmfels: *Combinatorial Commutative Algebra*, Springer Graduate Texts, New York, 2005.
18. G. Pataki: The geometry of cone-LP's, in [26].
19. P. Pragacz: Enumerative geometry of degeneracy loci, *Ann. Scient. Ec. Norm. Sup.* **4**. **21** (1988) 413–454.
20. D. Speyer: Horn's problem, Vinnikov curves, and the hive cone, *Duke Mathematical Journal* **127** (2005) 395–427.
21. J.F. Sturm: SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optimization Methods and Software* **11&12** (1999) 625–653.
22. E. Tevelev: *Projective Duality and Homogeneous Spaces*, Encyclopaedia of Mathematical Sciences, Springer Verlag, Berlin, 2005.
23. L. Vandenberghe and S. Boyd: Semidefinite programming, *SIAM Review* **38** (1996) 49–95.
24. V. Vinnikov: Self-adjoint determinantal representation of real plane curves, *Mathematische Annalen* **296** (1993) 453–479.
25. H.C. von Bothmer and K. Ranestad: A general formula for the algebraic degree in semidefinite programming. <http://arxiv.org/abs/math/0701877>.
26. H. Wolkowicz, R. Saigal and L. Vandenberghe: *Handbook of Semidefinite Programming*, Kluwer, 2000.