SDP and SOS

Eight Exercises for Pablo Parrilo's Lectures
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1. Consider a univariate polynomial of degree d, that is bounded by one in absolute value on the interval $[-1, 1]$. How large can its leading coefficient be? Give an SOS formulation for this problem, and solve it numerically for $d = 2, 3, 4, 5$. Can you guess what the general solution is as a function of d? Can you characterize the optimal polynomial?

2. Consider a given univariate rational function $r(x)$, for which we want to find a good polynomial approximation $p(x)$ of fixed degree d on the interval $[-2, 2]$.

   (a) Write an SOS formulation to compute the best polynomial approximation of $r(x)$ in the supremum norm.

   (b) Same as before, but now $p(x)$ is also required to be convex.

   (c) Same as before, but $p(x)$ is required to be a convex lower bound of $r(x)$ (i.e., $p(x) \le r(x)$ for all $x \in [-2, 2]$).

   (d) Let $r(x) = \frac{1-2x+x^2}{1+x+x^2}$. Find the solution of the previous subproblems (for $d = 4$), and plot them.

3. Given a set $\mathcal{S} \subseteq \mathbb{R}^n$ that strictly contains the origin, we define the *polar set* $\mathcal{S}^o \subseteq \mathbb{R}^n$ as:

$$\mathcal{S}^o := \{y \in \mathbb{R}^n \,|\, y^T x \le 1, \quad \forall x \in \mathcal{S}\}.$$

   (a) Let $\mathcal{S}$ be the feasible set of an SDP, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^m \,|\, \sum_{i=1}^{m} x_i A_i \preceq C\},$$

   where A_i and C are given symmetric matrices, and $C \succ 0$. Find a convenient description of $\mathcal{S}^o$. Can you optimize a linear function over $\mathcal{S}^o$?

4. Consider the polynomial $p(x) = x^4 + 2ax^2 + b$. For what values of (a, b) is this polynomial nonnegative? Draw the region of nonnegativity in the (a, b) plane. Where does the discriminant of p vanish? How do you explain this?

5. Consider the polynomial system $\{x + y^3 = 2, x^2 + y^2 = 1\}$.

   (a) Is it feasible over $\mathbb{C}$? How many solutions are there?

   (b) Is it feasible over $\mathbb{R}$? If not, give a Positivstellensatz-based infeasibility certificate of this fact.

6. In general, the SOS decomposition of a univariate polynomial is not unique. Given a specific basis of $\mathbb{R}[x]$ (for instance, the standard monomial basis we have been considering), a "natural" choice can be obtained by finding a matrix Q satisfying

$$p(x) = [x]_d^T \, Q \, [x]_d, \qquad Q \succeq 0$$

and such that the determinant of Q is as large as possible. This "central solution" can be computed by solving a convex optimization problem, since $\log\det Q$ is a concave function on the region where Q is positive semidefinite. In fact, this convex optimization problem can be reformulated a semidefinite programming problem.

   (a) Compute numerically the central solution for the polynomial $p(x) = x^6 - 6x^5 + 16x^4 - 24x^3 + 22x^2 - 12x + 4$.

   (b) Show, using the KKT optimality conditions, that in general the inverse of the optimal matrix Q is a Hankel matrix.

7. Consider a random variable X, with an unknown probability distribution supported on the set Ω, and for which we know its first $d+1$ moments $(\mu_0, \ldots, \mu_d)$. We want to find bounds on the probability of an event $S \subseteq \Omega$, i.e., want to bound $P(X \in S)$. We assume S and Ω are given intervals. Consider the following optimization problem in the decision variables c_k:

$$\min \sum_{k=0}^{d} c_k \mu_k \qquad \text{s.t.} \qquad \begin{cases} \sum_{k=0}^{d} c_k x^k \geq 1 & \forall x \in S \\ \sum_{k=0}^{d} c_k x^k \geq 0 & \forall x \in \Omega. \end{cases} \tag{1}$$

   (a) Show that any feasible solution of (1) gives a valid upper bound on $P(X \in S)$. How would you solve this problem?

   (b) Assume that $\Omega = [0, 5]$, $S = [4, 5]$, and we know that the mean and variance of X are equal to 1 and 1/2, respectively. Give upper and lower bounds on $P(X \in S)$. Are these bounds tight? Can you find the worst-case distributions?

8. Consider linear maps between symmetric matrices, i.e., of the form $\Lambda : \mathcal{S}^n \to \mathcal{S}^m$. A map is said to be a *positive map* if it maps the PSD cone S_+^n into the PSD cone S_+^m (i.e., it preserves positive semidefinite matrices).

   (a) Show that any linear map of the form $A \mapsto \sum_i P_i^T A P_i$, where $P_i \in \mathbb{R}^{n \times m}$, is positive. These maps are known as *decomposable* maps.

   (b) Show that the linear map $C : \mathcal{S}^3 \to \mathcal{S}^3$ (due to M.-D. Choi) given by:

$$C : A \mapsto \begin{bmatrix} 2a_{11} + a_{22} & 0 & 0 \\ 0 & 2a_{22} + a_{33} & 0 \\ 0 & 0 & 2a_{33} + a_{11} \end{bmatrix} - A.$$

   is a positive map, but is not decomposable.

Hint: Consider the polynomial defined by $p(x, y) := y^T \Lambda(xx^T) y$. How can you express positivity and decomposability of the linear map Λ in terms of the polynomial p?