EXERCISES ON LIFTS OF POLYTOPES

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1. Lecture 1

The first two exercises lead you through specialized/ad hoc constructions of polytopal lifts for two classes of polytopes that arise in combinatorial optimization. Both constructions are due to Yannakakis.

Problem 1. The parity polytope $PP_n$ in dimension $n$ is defined to be the convex hull of all vectors $v \in \{0,1\}^n$ with an odd number of 1s. The goal of this exercise is to prove that $PP_n$ has a small lift.

(a) Prove that the following inequalities cut out $PP_n$:

\[ 0 \leq x_i \leq 1 \quad \forall \quad i \in [n] \]
\[ \sum_{i \in A} x_i - \sum_{i \not\in A} x_i \leq |A| - 1 \quad \forall \quad A \subseteq [n], \quad |A| \text{ even} \]

Which of these inequalities are guaranteed to be facets of $PP_n$?

(b) For $k \in [n]$ odd, let $S_k$ be the convex hull of all 0/1 vectors in $\mathbb{R}^n$ with $k$ 1s. Prove that $x \in PP_n$ if and only if $x = \sum_{k \text{ odd}} \alpha_k y_k$ where $y_k \in S_k$ and $\sum \alpha_k = 1$, $\alpha_k \geq 0$ for all $k$.

(c) Prove that $PP_n$ is the projection onto the $x$-coordinates of the polytope described by the constraints:

\[ \sum_{k \text{ odd}} \alpha_k = 1, \quad x_i = \sum_{k \text{ odd}} z_{ik} \quad \forall \quad i \in [n], \]
\[ \sum_i z_{ik} = k\alpha_k \quad \forall \quad (\text{odd}) \quad k, \quad 0 \leq z_{ik} \leq \alpha_k \quad \forall \quad i, k \]

(d) Determine the size of this lift.

Problem 2. Let $G = ([n], E)$ be a graph with vertex set $[n]$ and edge set $E$. A set $S \subseteq [n]$ is stable if for every $i, j \in S$, $\{i, j\} \not\in E$. Let $\chi^S \in \{0,1\}^n$ be the characteristic vector of the stable set $S$. The stable set polytope, $STAB(G)$ is the convex hull of $\{\chi^S : S \text{ stable in } G\}$. No complete description of $STAB(G)$ is known and in general, optimizing over $STAB(G)$ is NP-hard.
(a) Check that the following inequalities are all valid on $STAB(G)$:

1. $0 \leq x_i \leq 1 \ \forall \ i \in [n]$
2. $x_i + x_j \leq 1 \ \forall \ \{i, j\} \in E$
3. $\sum_{i \in C} x_i \leq (|C| - 1)/2 \ \forall \ C$ odd cycle in $G$

**Definition.** The polytope in $\mathbb{R}^n$ cut out by inequalities (1)-(2) is called the fractional stable set polytope of $G$ and is denoted by $FRAC(G)$.

**Definition.** The polytope in $\mathbb{R}^n$ cut out by the inequalities (1)-(3) is called the odd cycle polytope of $G$ and is denoted as $ODDCYC(G)$.

Even though there may be exponentially many inequalities of type (3), one can optimize over $ODDCYC(G)$ in polynomial time (in $n$) using the ellipsoid algorithm for linear programming. It turns out that we can also prove the same result by exhibiting a small polytopal lift of $ODDCYC(G)$ as follows.

(b) Pick an $x \in FRAC(G)$ and define $l_{ij} := 1 - x_i - x_j$ for each edge $\{i, j\} \in E$. Since $l_{ij} \geq 0$ for all $\{i, j\} \in E$, interpret $l_{ij}$ as the length of edge $\{i, j\}$. Show that a constraint of type (3) is violated by $x$ if and only if the shortest odd cycle in $G$ has length less than one.

(c) For every pair of nodes $i$ and $j$, let $e_{ij}$ and $o_{ij}$ denote the even and odd distances (length of shortest paths) between nodes $i$ and $j$. Note that $e_{ii} = 0$ but $o_{ii}$ may be non-trivial and so we discard the variables $e_{ii}$. Consider the polytope $Q(G)$ described by the inequalities:

4. $0 \leq x_i \leq 1 \ \forall \ i \in [n]$
5. $0 \leq o_{ij} \leq 1 - x_i - x_j \ \forall \ \{i, j\} \in E$
6. $o_{ij} \leq o_{ik} + e_{kj}, \ e_{ij} \leq o_{ik} + o_{kj} \ \forall \ \{i, k\} \in E, \ j \in [n]$
7. $o_{ii} \geq 1 \ \forall \ i \in [n]$

Prove that $ODDCYC(G)$ is the projection of $Q(G)$ onto the $x$-variables. What is the size of $Q(G)$?

**Problem 3.** Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polytope. A set $Q \subseteq \mathbb{R}^n \times \mathbb{R}_+^m$ is called a lift of $P$ if $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^m : Rx + Sy = t\}$ and $P = \pi_{\mathbb{R}^n}(Q)$. It can be argued that there is no loss of generality in considering only lifts of $P$ of this form. Note that such a lift is “small” if $m$ is “small”. If $A \in \mathbb{R}^{f \times n}$ then $P$ always has the trivial lift

$$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^f : Ax + Iy = b\}$$

with $n + f$ variables, $n + f$ equations and $f$ inequalities. This lift is small only if $f$ is small.
Can you find lifts of the following polytopes with \( m < f \)?

(a) The square \([-1, 1]^2\) and more generally, the unit cube \([-1, 1]^n\)?
(b) The standard cross-polytopes in \(\mathbb{R}^n\) (dual to the cubes in (a))?  
(c) A pentagon? A hexagon?

2. Lecture 2

Problem 4. Let \( P = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_1 + x_2 \geq \frac{1}{2}\} \). Compute \( N(P) \) and decide whether it equals the convex hull of integer points in \( P \).

Problem 5. Lovász and Schrijver proposed a second hierarchy of relaxations for computing the integer hull of a polytope \( P \subseteq [0, 1]^n \) where the lifts are spectrahedra. This exercise explores this semidefinite programming hierarchy.

(a) Let \( L \) be a lattice and \( f \in \mathbb{R}^L \). Define two \(|L| \times |L|\) matrices \( W^f \) and \( D^f \) as follows:

\[
W^f := (w_{i,j})_{i,j \in L}, \quad w_{i,j} := f(i \lor j) \\
D^f := \text{diag}(f(i))_{i \in L}.
\]

Prove that if \( f = Zg \), where \( Z \) is the Zeta matrix of \( L \), then \( W^f = ZD^gZ^T \).

(b) Let \( H \) be the cone spanned by the columns of \( Z \). Prove that \( f \in H \) if and only if \( W^f \succeq 0 \).

(c) As in the lecture, let \( P \subseteq [0, 1]^n \) be a polytope, \( K \) be the cone over \( \{1\} \times P \) and \( K^0 \) be the cone generated by \( \{1\} \times (P \cap \{0, 1\}^n) \). Define

\[
M_+(K) := \left\{ Y \in \mathbb{R}^{(n+1) \times (n+1)} : \begin{array}{l}
(i) \ Y \succeq 0 \\
(ii) \ Ye_0 = \text{diag}(Y) \\
(iii) \ Ye_j, Ye_0 - e_j \in K \ \forall \ j \in [n]
\end{array} \right\}
\]

and

\[
N_+(K) := \{ \text{diag}(Y) : Y \in M_+(K) \}.
\]

Explain why \( M_+(K) \) is also a relaxation of \( \text{cone}(\mathcal{F}) \) where \( \mathcal{F} \) is the set of supports of 0/1-vectors in \( P \).

(d) Prove that \( K^0 \subseteq N_+(K) \subseteq N(K) \subseteq K \).

As for the \( N \)-operator we can define \( N^1_+(K) = N_+(K) \) and \( N^i_+(K) = N_+(N^{i-1}_+(K)) \). By (d), we have \( N^n_+(K) = K^0 \).

Problem 6. Is \( N_+(P) \) strictly contained in \( N(P) \) for the polytope \( P \) in Problem 4?
Problem 7. Compute the theta bodies, and the spectrahedra that they are projections of, of the ideal $\langle (x+1)x(x-1)(x-2) \rangle$. Draw pictures if you can.

Problem 8. Consider an odd cycle $G$ with $2k + 1$ vertices and $k \geq 2$. Argue that $I_G$ is not $TH_1$-exact. It is known that $STAB(G)$ is described by the following additional inequality to $FRAC(G)$: $\sum x_i \leq k$. Show that $STAB(G) = TH_2(I_G)$. Perhaps try a 5-cycle or 7-cycle as a warm up.

Problem 9. Compute the first theta body of the vanishing ideal of $S = \{(0,0), (1,0), (0,1), (2,2)\}$.

Problem 10. Prove that $Q_k(I) \subseteq TH_k(I)$ for any ideal $I \subset \mathbb{R}[x_1, \ldots, x_n]$.