CLASSICAL INVARIANT THEORY AND THE VIRASORO ALGEBRA

Nolan R. Wallach

INTRODUCTION.

The Virasoro algebra (a certain central extension of the Lie algebra of vector fields on the circle) has come to play a more important role in such fields as quantum field theory and number theory. In [3], Wakimoto and Yamada gave a new relationship with classical invariant theory. The purpose of this note is to give a more direct relationship between highest weight vectors for the Virasoro algebra and characters of the unitary groups U(n), $n \geq 1$. In particular, our result gives (in principle) a method of calculating all highest weight vectors for all Verma modules for the Virasoro algebra. However, the combinatorial problems in the general case are very difficult. In the special case studied in [3], we give a simple proof of the main result announced in [3].

We thank Alvany Rocha for kindly sending us a copy of the announcement in [3]. The representations in Section 2 appear for the first time in [1].

1. A REALIZATION OF THE WITT ALGEBRA

We set $M_n(\mathbb{C})$ equal to the space of $n \times n$ matrices over $\mathbb{C}$. We look upon $GL(n,\mathbb{C})$ and $U(n)$ as subsets of $M_n(\mathbb{C})$. Let $\underline{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$. If $X \in M_n(\mathbb{C})$ then we write

$$X = X_1 + iX_2$$

with $X_1, X_2 \in \underline{u}(n)$.

We define for each $m \in \mathbb{Z}$ the vector field d_m on U(n) as follows:

Vertex Operators in Mathematics and Physics - Proceedings of a Conference November 10-17, 1983. Publications of the Mathematical Sciences Research Institute #3, Springer-Verlag, 1984.

$$d_m f(g) = \frac{d}{dt}\Big|_{t=0} \{f(ge^{t(g^m)_1}) + if(ge^{t(g^m)_2})\}.$$

If f extends to a holomorphic function on $GL(n,\mathbb{C})$ then

$$d_m f(g) = \frac{d}{dt}\Big|_{t=0} f(ge^{tg^m}).$$

One checks that

$$[d_p, d_q] = (q-p)d_{p+q} \,. \tag{1}$$

We set $\underline{d} = \oplus \mathbb{C} d_p$. Then $\underline{d}$ is isomorphic to the Witt algebra. We note that if $n=1$ then

$$d_m = \frac{1}{i} e^{im\theta} \frac{d}{d\theta},$$

the usual basis of the Witt algebra.

If M is an oriented manifold with volume form ω and if D is a differential operator on M then we define D^T by

$$\int_M (Df) g\omega = \int_M f(D^T g)\omega$$

for, say, $f,g \in C_c^\infty(M)$. If X is a smooth vector field on M then

$$X^T = -X - \phi_{\omega,X}$$

with $\phi_{\omega,X}$ a smooth function on M which is up to sign the divergence of X relative to ω.

On U(n) we take ω to be the normalized invariant measure. We will write

$$\int_{U(n)} f\omega = \int_{U(n)} f(g)dg.$$

We also write ϕ_m for ϕ_{ω,d_m} .

Then we have

(2)
$$\phi_m(g) = \sum_{k=0}^{m-1} tr(g^{k+1})tr(g^{m-k-1}) \qquad \text{if } m \geq 0$$
$$\phi_{-m}(g) = \sum_{k=0}^{m-1} tr(g^{-m+k+1})tr(g^{-k-1}) \quad \text{if } m > 0.$$

The operators d_m are not left invariant on U(n); however, they are central operators. They are intimately connected with the "symbolic operators" of classical invariant theory. In fact, let us set $S_k(g) = tr(g^k)$. It is easy to check that

(3)
$$d_m S_k = kS_{k+m} .$$

We define by $\lambda,\mu \in \mathbb{C}$ the operators $\sigma_{\lambda,\mu}(d_m)$ by

$$\sigma_{\lambda,\mu}(d_m)f = d_m f + (\lambda+m\mu)S_m f.$$

Then one checks (using (3))

(4)
$$[\sigma_{\lambda,\mu}(d_p), \sigma_{\lambda,\mu}(d_q)] = (q-p)\sigma_{\lambda,\mu}(d_{p+q}).$$

We note that

(5)
$$\sigma_{k,0}(d_m)\cdot \det^{-k} = 0 \text{ for all } m \in \mathbb{Z}.$$

Indeed, $(d_m \det^{-k})(g) = \frac{d}{dt}\Big|_{t=0} \det(ge^{tg^m})^{-k}$

$$= \frac{d}{dt}\Big|_{t=0} \det(g)^{-k}e^{-k \, tr \, g^m}$$

$$= -k \, tr(g^m)\det(g)^{-k}.$$

We will see in Section 3 that this observation implies the result of [3].

2. FOCK REALIZATIONS OF HIGHEST WEIGHT MODULES FOR THE VIRASORO ALGEBRA

Let $\hat{\underline{d}}$ be the Lie algebra with basis $\{d_n \mid n \in \mathbb{Z}\}$ and κ a central element with commutation relations

(1) $$[d_m, d_n] = (n-m)d_{n+m} + \frac{1}{12}n(n^2-1)\delta_{m,-n}\kappa.$$

Let $V = \mathbb{C}[x_1, x_2, \ldots]$. We define operators on V as follows:

$$u(n)f = n\frac{\partial}{\partial x_n}f,\ n \geq 1$$

(2) $$u(0)f = 0$$

$$u(-n)f = x_n f,\ n \geq 1.$$

We set

$$\pi_0(d_{2n}) = \frac{1}{2}u(n)^2 - \sum_{k=1}^{\infty} u(n-k)u(n+k)$$

$$\pi_0(d_{2n+1}) = -\sum_{k=0}^{\infty} u(n-k)u(n+k+1).$$

Then one checks that if we set $\pi_0(\kappa) = I$ then (π_0, V) is a representation of the Virasoro algebra.

We note that

(3) $$[\pi_0(d_m), u(n)] = nu(n+m).$$

Thus $\pi_0(d_m)$ and $u(q)$ can be considered to be a simultaneous "quantization" of the operators d_m and multiplication by S_q. In light of (3) we can define for $\lambda, \mu \in \mathbb{C}$ the operators

(4) $$\pi_{\lambda,\mu}(d_m) = \pi_0(d_m) + (\lambda+m\mu)(u(m)) - (\frac{\lambda^2}{2} - \frac{\mu^2}{2})\delta_{m,0}I.$$

Then (3) implies

Lemma 2.1. If we set $\pi_{\lambda,\mu}(\kappa) = (1-12\mu^2)I$ then $(\pi_{\lambda,\mu},V)$ is a representation of the Virasoro algebra with

$$\pi_{\lambda,\mu}(d_0)\cdot 1 = -\left(\frac{\lambda^2}{2} - \frac{\mu^2}{2}\right)1.$$

On V we put the pre-Hilbert space structure

$$\langle x^I,x^J\rangle = \delta_{I,J}I!(1^{i_1}2^{i_2}\ldots) \tag{5}$$

Here we have used standard multi-index notation. That is, $x^I = x_1^{i_1}x_2^{i_2}\ldots$, $I! = i_1!i_2!\ldots$. We will also write $|I| = \Sigma i_j$ and $\langle I\rangle = \Sigma ji_j$. One checks that

$$\langle u(n)f,g\rangle = \langle f,u(-n)g\rangle \ . \tag{6}$$

Hence

$$\langle \pi_{\lambda,\mu}(d_n)f,g\rangle = \langle f,\pi_{\bar{\lambda},-\bar{\mu}}(d_{-n})g\rangle \ . \tag{7}$$

Thus if we define $d_n^* = d_{-n}$, $n \in \mathbb{Z}$ and $\kappa^* = \kappa$ and extend this operation to be conjugate linear we find that if $\hat{\underline{d}}_{\mathbb{R}} = \{d \in \hat{\underline{d}} \mid d^* = -d\}$ then if $\lambda \in \mathbb{R}$, $\mu \in i\mathbb{R}$

$$\langle \pi_{\lambda,\mu}(d)f,g\rangle = -\langle f,\pi_{\lambda,\mu}(d)g\rangle,\ d \in \hat{\underline{d}}_{\mathbb{R}}. \tag{8}$$

3. THE INTERTWINING OPERATORS

For $g \in U(n)$, $x_i \in \mathbb{C}$, $\Sigma|x_i| < \infty$ let $\gamma(g,x) = \sqrt{2}\sum_{m=1}^{\infty} x_m tr(g^m)/m$. Then $\gamma(g,x)$ is a real analytic function on U(n).

If $f \in C^\infty(U(n))$ we define

$$T_n(f)(x) = \int_{U(n)} e^{\gamma(g,x)} f(g)dg.$$

We note that if $f(e^{i\theta}g) = e^{-im\theta}f(g)$, $\theta \in \mathbb{R}$, $g \in U(n)$, then $T_n(f)(x)$ is a polynomial in $x_1, x_2, \ldots, x_m$ of degree at most m.

Theorem 3.1. If $m > 0$ then

$$\pi_{\lambda,\mu}(d_m)T_n(f) = T_n(\sigma_{\sqrt{2}\lambda+n,\sqrt{2}\mu}(d_m)f)$$

for all $\lambda,\mu \in \mathbb{C}$, $f \in C^\infty(U(n))$.

Proof. Since d_1, d_2 generate $\sum_{m\geq 1} \mathbb{C}d_m$ as a Lie algebra it is enough to check the formula for d_1, d_2. In each case this is a straightforward calculation which we now do.

$$\pi_{\lambda,\mu}(d_1) = (\lambda+\mu)\frac{\partial}{\partial x_1} - \sum_{m=1}^{\infty} (m+1)x_m \frac{\partial}{\partial x_{m+1}}.$$

Thus

$$(\pi_{\lambda,\mu}(d_1)T_n(f))(x) = \sqrt{2}(\lambda+\mu)\int_{U(n)} tr(g)e^{\gamma(g,x)}f(g)dg$$

$$- \sqrt{2} \sum_{m=1}^{\infty} x_m \int_{U(n)} tr(g^{m+1})e^{\gamma(g,x)}f(g)dg$$

$$= \sqrt{2}(\lambda+\mu)\int_{U(n)} tr(g)e^{\gamma(g,x)}f(g)dg$$

$$-\int_{U(n)} (d_1 \cdot e^{\gamma(g,x)})f(g)dg$$

$$= \sqrt{2}(\lambda+\mu)\int_{U(n)} tr(g)e^{\gamma(g,x)}f(g)dg$$

$$-\int_{U(n)} e^{\gamma(g,x)} d_1^T f(g)dg.$$

Since $d_1^T = -d_1 - n\,tr(g)$ (formula 1 (2)) the equation is true for d_1. We now look at d_2. $\pi_{\lambda,\mu}(d_2) = (2\pi+4\mu)\frac{\partial}{\partial x_2}$

$$\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - \sum_{m=1}^{\infty}(m+2)x_m\frac{\partial}{\partial x_{m+2}}.$$

Thus

$$\pi_{\lambda,\mu}(d_2)T_n(f)(x) = \sqrt{2}(\lambda+2\mu)\int_{U(n)} tr(g^2)e^{\gamma(g,x)}f(g)dg$$

$$-\int_{U(n)} (tr(g))^2 e^{\gamma(g,x)}f(g)dg - \sum_{m=1}^{\infty}\sqrt{2}x_m\int_{U(n)} tr(g^{m+2})e^{\gamma(g,x)}f(g)dg$$

$$= \sqrt{2}(\lambda+2\mu)\int_{U(n)} tr(g^2)e^{\gamma(g,x)}f(g)dg$$

$$-\int_{U(n)} (tr\ g)^2 e^{\gamma(g,x)}f(g)dg - \int_{U(n)} (d_2 e^{\gamma(g,x)})f(g)dg.$$

This time $d_2^T = -d_2 - tr(g)^2 - n\,tr(g^2)$. The formula now follows for d_2.

We combine this result with formula 1 (6) to obtain

<u>Corollary</u> 3.2. $\pi_{\frac{p-n}{\sqrt{2}}}(d_m)T_n(\det^{-p}) = 0$ for $p \geqslant 0$, $n \geqslant 1$, $m \geqslant 1$.

We note that the coefficient of x_1^{np} in $T_n(\det^{-p})$ is

$$\frac{2^{\frac{np}{2}}}{(np)!}\int_{U(n)} tr(g)^{np}\det(g)^{-p}dg = 2^{np/2}/(np)!\ .$$

We can give more explicit formulas for the action of T_n using classical results of Frobenius. We parametrize the characters of the finite dimensional irreducible polynomial representations of U(n) by $\Lambda = (m_1,\dots,m_n)$, $m_1 \geqslant m_2 \geqslant \dots \geqslant m_n \geqslant 0$, $m_i \in \mathbb{Z}$. Let χ_Λ be the corresponding character. We also look upon Λ as a partition of $|\Lambda| = m_1+\ldots+m_n$. Let χ^Λ be the corresponding character of $S_{|\Lambda|}$, the symmetric group on $|\Lambda|$ letters. If $I = (i_1,i_2,\ldots)$ and if $\langle I\rangle = \Sigma j i_j = |\Lambda|$ we denote by C_I the conjugacy class in $S_{|\Lambda|}$ which corresponds to i_1 cycles of length 1, i_2 cycles of length 2, ... Let $|C_I|$ be the order of C_I. Then the Weyl character formula implies (see [2], p. 67, 5.2; 8 and its derivation)

$$T_n(\bar{x}_\Lambda) = \frac{2^{|\Lambda|/2}}{|\Lambda|!} \sum_{\langle I\rangle = |\Lambda|} |C_I|\,\chi^\Lambda(C_I)x^I.$$

We note that if we set $\phi_{n,p} = T_n(\det^{-p})$ then we have

<u>Lemma</u> 3.3. $$\int_{SU(n)} e^{\sqrt{2}\Sigma t^m x_m \operatorname{tr}(g^m)/m} dg = \sum_{p=0}^{\infty} t^{np}\phi_{n,p}(x).$$

In particular this gives $e^{\sqrt{2}\Sigma t^m x_m/m} = \sum_{p=0}^{\infty} t^p \phi_{1,p}(x)$.

Set $V_n = \{f \in V \mid \pi_0(d_0)f = nf\}$. That is $V_n = \sum_{\langle I\rangle = n} \mathbb{C}x^I$.

Set $I_n = \sum_{|\Lambda|=n} \mathbb{C}\bar{x}_\Lambda$. Then the above formula implies that

$$T_n\colon I_n \longrightarrow V_n$$

is bijective. Hence in principle one can compute all of the highest weight vectors for all of the $\pi_{\lambda,\mu}$ using Theorem 3.1.

REFERENCES

[1] R. Goodman and N.R. Wallach, Projective unitary positive energy representations of $\mathrm{Diff}(S^1)$, preprint.

[2] D.E. Littlewood, <u>The Theory of Group Characters</u>, Oxford University Press, London, 1940.

[3] M. Wakimoto and H. Yamada, Irreducible decompositions of Fock representations of the Virasoro algebra, preprint.

Rutgers University
New Brunswick, NJ 08903