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ON MATSUSHIMA'S FORMULA FOR THE BETTI NUMBERS OF A LOCALLY SYMMETRIC SPACE

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1. Introduction. Let G be a connected semi-simple Lie group with finite center and with no connected normal, compact subgroups. Let $K\subset G$ be a maximal compact subgroup and let $\Gamma\subset G$ be a discrete subgroup acting freely on $X=G/K$ and so that $\Gamma\backslash G$ is compact. Let $M=\Gamma\backslash X$ then M is a typical compact locally symmetric space of negative curvature. Let G act by the right regular action, π_Γ, on $L^2(\Gamma\backslash G)$. In Matsushima [13] a formula for the Betti numbers of M is given in terms of the multiplicities of certain unitary representations of G in π_Γ. In this paper we investigate the existence of the representations of G that come into the Matsushima formula. In particular we show that if X is irreducible and Hermitian symmetric and if rank $X>p$ then there are no unitary representations of G that satisfy the conditions for the Matsushima formulas for the $(0, p)$ Betti number of $\Gamma\backslash X$, $b_{0,p}(\Gamma\backslash X)$. Thus we find that if $p<$rank X, $b_{0,p}(\Gamma\backslash X)=0$. In particular if rank $X>1$ we recover Matsushima's theorem [12] that the first Betti number of $\Gamma\backslash X$ is zero.

Actually this result (on the first Betti number) follows from the more general theorem of Kazhdan which says if G is a simple Lie group of split rank larger than 1 and if $\Gamma\subset G$ is a discrete subgroup so that $\Gamma\backslash G$ has finite volume relative to some Haar measure on G then $\Gamma/[\Gamma, \Gamma]$ is finite.

In light of the above result of Kazhdan it is reasonable to study the unitary representations that appear in the formula of Matsushima for the first Betti number in the case G has split rank 1. We show that in this case such representations always exist. We show that there are at most two such unitary representations and if G is locally $SO(n, 1)$ there is exactly one, let us call it π_1 (see Lemma 2.1, Prop. 2.2 and Lemma 4.4). This gives us some interesting examples. E. B. Vinberg [16] has constructed a uniform discrete subgroup $\Gamma\subset SO_e(n, 1)$ for $3\leq n\leq 5$, which is arithmetic in the sense of Borel, Harish-Chandra, such that $\Gamma/[\Gamma, \Gamma]$ is infinite. In Johnson-Wallach [7] it is shown that π_1 cannot be tempered in the sense of Harish-Chandra (see G. Warner [14]) if $n\geq 4$. Thus there exists an arithmetic uniform discrete subgroup Γ of a simple algebraic group G and a non-trivial non-tempered representation with positive multiplicity in $L^2(\Gamma\backslash G)$. This fact would require furthur refinement about so-called "generalized Ramanujan conjecture". We are grateful to Professor A. Borel for kindly pointing out this paper of Vinberg.

We conclude the paper by giving the representations that occur in the Matsushima formula for the (p, q) Betti numbers in the case of $SU(2, 1)$ and study the implications of Riemann-Roch and Gauss-Bonnet theorem in this case.

2. The existence and uniqueness of certain representations

In this section we study the case where G is a connected, simple, Lie group with finite center having split rank 1. Let $K\subset G$ be a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively the Lie algebras of G and K. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. We denote by Ad the adjoint representation of G on $\mathfrak{g}_C$ (the complexification of $\mathfrak{g}$). Let $(\tau, \mathfrak{p}_C)$ be the representation of K on $\mathfrak{p}_C$ (the complexification of $\mathfrak{p}$) given by $\mathrm{Ad}|_K$. There are two possibilities:

(1) $(\tau, \mathfrak{p}_C)$ is reducible and $\mathfrak{p}_C=\mathfrak{p}^+\oplus\mathfrak{p}^-$ with $(\tau, \mathfrak{p}^+)$ and $(\tau, \mathfrak{p}^-)$ irreducible (in this case G is locally isomorphic with $SU(n, 1)$ for some n).

(2) $(\tau, \mathfrak{p}_C)$ is irreducible.

In case (1) we take τ_0 to be $(\tau, \mathfrak{p}^-)$ in case (2) we take τ_0 to be $(\tau, \mathfrak{p}_C)$.

Let Ω be the Casimir operator of $\mathfrak{g}$. That is, if B is the Killing form of $\mathfrak{g}$, if $X_1, \cdots, X_n$ is a basis of $\mathfrak{g}$ and if $Y_1, \cdots, Y_n$ satisfy $B(X_i, Y_j)=\delta_{ij}$ then $\Omega=\sum X_i Y_i$.

Lemma 2.1. *There exists an irreducible unitary representation (π_1, H_1) of G so that $(\pi_1|_K: \tau_0)=1$ and $\pi_1(\Omega)=0$. (Here if σ_1, σ_2 are representations of K then $(\sigma_1: \sigma_2)$ is the dimension of the space of K-intertwining operators from σ_1 to σ_2).*

Proof. In Kostant [10] (see also Johnson-Wallach [7]) it is shown that if π is a (non-unitary) prinicipal series representation of G and if σ is an irreducible unitary representation of K then $(\pi|_K: \sigma)=1$ or 0 depending on whether or not σ has an M-fixed vector. In Johnson-Wallach [7] it is shown if (π^0, V) is the representation of G on the space V of K-finite, C^∞ functions on G/MAN ($MAN=P$ a minimal parabolic subgroup of G, $M=K\cap P$, $K/M=G/P$) defined by

$$(\pi^0(X)f)(x)=\frac{d}{dt}f(\exp(-tX)x)|_{t=0} \text{ for } f\in C^\infty(G/P)$$

then every constituent of the (finite) composition series extends to a unitary representation of G. Now τ_0 has an M-fixed vector thus $(\pi^0|_K: \tau_0)=1$. Hence exactly one of the irreducible constituents of π^0 contains τ_0. We denote this constituent by π_1. Clearly $\pi^0(\Omega)=0$ $(1\in V)$. Thus $\pi_1(\Omega)=0$.

Proposition 2.2. *Suppose that G is in case* (1) *above or that G is locally* *isomorphic with* $SO_e(n, 1)$, $n \geq 3$ (*the identity component of the group of the quadratic form* $\sum_{i=1}^{n} x_i^2 - x_{n+1}^2$). *If* (π, H) *is an irreducible unitary representation of* G *so that* $(\pi|_K : \tau_0) \neq 0$ *and* $\pi(\Omega) = 0$ *then* π *is unitarily equivalent with* π_1.

Proof. The assertion in case (1) will be proved in the next section. We therefore assume that G is locally isomorphic with $SO_e(n, 1) = G_0$. If (π, H) contains τ_0 then since the center of G, Z, is contained in K and $\tau_0(Z) = \{I\}$ we see that $\pi(Z) = I$. We may therefore assume that $G = G_0$. Let $P = MAN$ be a minimal parabolic subgroup of G such that $M = P \cap K$ and the Lie algebra of A, $\mathfrak{a}$, is contained in $\mathfrak{p}$. Let $\hat{M}$ be the set of equivalence classes of irreducible unitary representations of M and let $\mathfrak{a}^*_C$ be the space of complex valued linear forms on $\mathfrak{a}$. If $\mathfrak{n}$ is the Lie algebra of N then there is a unique element, $H \in \mathfrak{a}$, so that $\mathrm{ad}H|_{\mathfrak{n}} = I$. Now $\mathfrak{a} = \boldsymbol{R}H$. If $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*_C$ let $(\pi_{\xi,\nu}, H^{\xi,\nu})$ be the corresponding (non-unitary) prinicipal series representation of G_0. That is $H^{\xi,\nu}$ is the space of all $f: G \to H_\xi$ ((ξ, H_ξ) is a representative of ξ) such that f is measurable and

(i) $f(gman) = \xi(m)^{-1} e^{-\nu(\log a)} f(g)$

for $g \in G$, $m \in M$, $a \in A$, $n \in N$ and log: $A \to \mathfrak{a}$ is the inverse of exp: $\mathfrak{a} \to A$

(ii) $\int_K \|f(k)\|^2 dk < \infty$

(iii) $(\pi_{\xi,\nu}(g)f)(x) = f(g^{-1}x)$

We note that in the notation of Lemma 2.1 $\pi^0 = \pi_{1,0}$, 1 the class of the trivial representation of M.

The subquotient theorem (Harish-Chandra [1]) says that (π, H) is infinitesimally equivalent with a constituent of the composition series of $\pi_{\xi,\nu}$ for some $\xi \in \hat{M}$, $\nu \in \mathfrak{a}^*_C$. Frobenius reciprocity implies that since $(\pi|_K : \tau_0) \neq 0$ then if π is equivalent to a constituent of $\pi_{\xi,\nu}$, $(\tau_0|_M : \xi) \neq 0$.

Now K acts on $\mathfrak{p}$ as $SO(n)$ on $\boldsymbol{R}^n$. M is just $SO(n-1)$ imbedded in K as

$$\begin{bmatrix} SO(n-1) & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus if ξ_0 is the standard representation of M on $\boldsymbol{C}^{n-1}$ and 1 is the trivial representation of M, $\tau_0|_M = \xi_0 \oplus 1$. Thus π must be a constituent of $\pi_{1,\nu}$ for some ν or $\pi_{\xi_0,\nu}$ for some ν. If $\pi_{1,\nu}(\Omega) = 0$ then $\nu(H) = 0$ or $n-1$. Since $\pi_{1,0}$ and $\pi_{1,n-1}$ have the same composition series we see that if π is a constituent of $\pi_{1,\nu}$ for some ν then π is equivalent to π_1.

If $\pi_{\xi_0,\nu}(\Omega) = 0$ then $\nu(H) = z$ or $n-z-1$ for some $z \in \boldsymbol{C}$ by the usual formula for $\pi_{\xi_0,\nu}(\Omega)$:

$$\pi_{\xi_0,\nu}(\Omega) = \text{const. } \{\nu(H)^2 - (n-1)\nu(H) + c(\xi_0)\},$$

where $c(\xi_0) \in \boldsymbol{C}$ is a constant depending only on ξ_0. $\pi_{\xi_0,n-z-1}$ and $\pi_{\xi_0,z}$ have the same character hence the same composition series. Thus since $(\tau_0|_M : \xi_0) = 1$

there is at most one other irreducible unitary representation of G satisfying the required conditions and it must appear in $\pi_{\xi_0,z}$. We show that the constituent of $\pi_{\xi_0,z}$ containing τ_0 is infinitesimally equivalent to π_1 and thereby prove the proposition.

Let T_1 be a maximal torus of M. Then T_1A is a Cartan subgroup of G_0. Let $G_C=SO(n+1, \boldsymbol{C})$ and let $C\subset G_C$ be the Cartan subgroup corresponding to T_1A. Let $\mathfrak{h}$ be the Lie algebra of C in $\mathfrak{g}_C$. Let Δ be the root system of $(\mathfrak{g}_C, \mathfrak{h})$ and let Δ^+ be a system of positive roots such that if $\alpha\in\Delta$ and $\alpha(H)>0$ then $\alpha>0$. Then if $\mathfrak{n}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{n}^+\cap\mathfrak{g}=\mathfrak{n}$. Let $\alpha_1, \cdots, \alpha_l$ be the corresponding simple roots (l is the largest integer $\leq(n+1)/2$). Let $\mathfrak{m}_C$ be the complexification of the Lie algebra of M. Set $\Delta_M=\{\alpha\in\Delta\,|\,\mathfrak{g}_\alpha\subset\mathfrak{m}_C\}$. Then by relabeling the $\alpha_1, \cdots, \alpha_l$ we may assume $\alpha_2, \cdots, \alpha_l\in\Delta_M$. Let $v_0\in H_{\xi_0}=\boldsymbol{C}^{n-1}$ be a non-zero highest weight vector and let w_0 be a lowest vector. If $f\in H^{\xi_0,\nu}$ set $S(f)(g)=(f(g), w_0)$ $((Z, W)=\sum Z_i\bar{W}_i)$. If $f\in C^\infty(G)$ define for $X\in\mathfrak{h}_C$, $X=X_1+iX_2$, $X_i\in\mathfrak{g}$, $(R_Xf)(g)=\dfrac{d}{dt}(f(g\exp tX_1)+if(g\exp tX_2))|_{t=0}$. If $\xi_0(t)v_0=\lambda(t)v_0$ for all $t\in T_1$ and if $f\in H^{\xi_0,\nu}$ then

(I) $R_XS(f)=0$ for $X\in\mathfrak{n}^-$

(II) $S(f)(gta)=\lambda(t)e^{-\nu(\log a)}S(f)(g)$ for $t\in T_1$, $a\in A$.

The Borel-Weil theorem (c.f. Wallach [17]) implies that S is a 1–1 map from the space of C^∞ elements of $H^{\xi_0,\nu}$, $H^{\xi_0,\nu}_\infty$, onto the space of all $f\in C^\infty(G)$ satisfying (I), (II). Let $X^{\lambda,\nu}$ be the space of all C^∞ functions on G satisfying (I), (II). If $f\in X^{\lambda,\nu}$, $x, g\in G$ set $(T_{\lambda,\nu}(g)f)(x)=f(g^{-1}x)$. Then $S\circ\pi_{\xi_0,\nu}(g)=T_{\lambda,\nu}(g)\circ S$. Hence the representations $(T_{\lambda,\nu}, X^{\lambda,\nu})$ and $(\pi_{\xi_0,\nu}, H^{\xi_0,\nu}_\infty)$ of $\mathfrak{g}$ are infinitesimally equivaldent.

Let now $E_{-\alpha_1}\in\mathfrak{g}_{-\alpha_1}-\{0\}$ $(\mathfrak{g}_{-\alpha_1}=\{X\in\mathfrak{g}_C\,|\,[h, X]=-\alpha_1(h)X$ for all $h\in\mathfrak{h}\})$. Set for $f\in H^{1,0}_\infty$, $B(f)(g)=(R_{E_{-\alpha_1}}f)(g)$. A direct computation gives $B(H^{1,0}_\infty)\subset X^{\lambda,\nu}$ with $\nu=\alpha_1|_{\mathfrak{a}}$. But B is clearly non-zero. Hence $\nu=z$ or $n-z-1$. Furthermore, it is shown in Johnson-Wallach [7] that $H^{1,0}$/(constants) is irreducible. Thus (π_1, H_1) is infinite simally equivalent with $(\pi_{1,0}, H^{1,0}$/constants) and by the above is contained in $\pi_{\xi_0,z}$. This completes the proof of the Proposition.

NOTES. 1. If G satisfies (2) then one can use the same sort of arguments as those proving Proposition 2.2 to show that there are at most 2 irreducible unitary representations, π, of G satisfying $(\pi|_K: \tau_0)\neq0$ and $\pi(\Omega)=0$.

2. We note that if π is irreducible $(\pi|_K: 1)\neq0$ and $\pi(\Omega)=0$ then π is the trivial representation of G. Indeed, the subquotient theorem implies π is a constituent of $\pi_{1,\nu}$ for some ν. But $\pi_{1,\nu}(\Omega)=0$ implies $\nu=0$ or $\nu=2\rho$. Both contain the trivial representation.

Using a similar argument we prove

Proposition 2.3. *Let G be locally isomorphic with $SU(n,1)$ and $n\geq 2$. Then G satisfies* (1). *As a representation space of K, $\mathfrak{p}^+\otimes\mathfrak{p}^-=\boldsymbol{C}v_0\oplus(\tau_{1,1}, V_{1,1})$ with $(\mathrm{Ad}(k)\otimes\mathrm{Ad}(k))(v_0)=v_0$, $k\in K$ and $(\tau_{1,1}, V_{1,1})$ is irreducible.*

(1) *There exists an irreducible unitary representation $(\pi^{1,1}, H_{1,1})$ of G so that $(\pi^{1,1}|_K: \tau_{1,1})\neq 0$ and $\pi^{1,1}(\Omega)=0$.*

(2) *If $n=2$ and if π is an irreducible unitary representation of G so that $(\pi|_K: \tau_{1,1})\neq 0$ and $\pi(\Omega)=0$ then π is unitarily equivalent with $\pi^{1,1}$. Furthermore $\pi^{1,1}$ is a non-integrable, squeare integrable representation of G.*

Proof. We assume (as in the proof of Proposition 2.2) that $G=SU(n, 1)$. The statements about $\mathfrak{p}^+\otimes\mathfrak{p}^-$ are standard. Again we look at the composition series of $\pi^{1,0}$ and find that there is a unique constituent, $\pi^{1,1}$ containing $\tau_{1,1}$. Since every constituent of $\pi_{1,0}$ is unitarizable (1) follows.

To prove (2) take $G=SU(2,1)$. It is proved in Johnson-Wallach [7] that $\pi^{1,1}$ is non-integrable discrete series. As a representation of M, $\tau_{1,1}=1\oplus\xi_1\oplus\xi_2$, ξ_i characters of M. MA is a Cartan subgroup of G. Ordering the roots as in the proof of Proposition 2.1 the simple roots that are not roots of M consists of α_1, α_2 (say). One finds $H^{1,0}_\infty/\mathrm{Ker}(R_{E-\alpha_1})+\mathrm{Ker}(R_{E-\alpha_2})$ is the representation space of $\pi^{1,1}$. The proof of uniqueness now follows as in the proof of Proposition 2.2.

NOTE. (2) above is actually true for all $SU(n, 1)$, $n\geq 2$ using a similar argument to the proof of (2) above. Here one must use $R^k_{E-\alpha_1}R_{E-\alpha_2}$ for an appropriate $k\geq 1$ to "pick up" the extra representation of M. These "extra intertwining operators" will be studied systematically in another paper.

3. Certain representations with highest weights

Let G be a connected, simply connected, simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ (see Helgason [3] chapters 3 and 8 for the pertinent definitions).

Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. We *assume* that $[\mathfrak{k}, \mathfrak{k}]\neq\mathfrak{k}$. Let $\mathfrak{h}_*$ be a maximal abelian subalgebra of $\mathfrak{k}$. If $\mathfrak{h}$ is the complexification of $\mathfrak{h}_*$ in $\mathfrak{g}_C$ then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_C$.Let Δ be the root system of $\mathfrak{g}_C$ relative to $\mathfrak{h}$. We may (and do) assume that $\mathfrak{k}=\boldsymbol{R}iH\oplus[\mathfrak{k}, \mathfrak{k}]$ with $\alpha(H)=\pm 1$ or 0 for $\alpha\in\Delta$. Let $\Delta_K=\{\alpha\in\Delta\,|\,\mathfrak{g}_\alpha\subset\mathfrak{k}_C\}$, $\Delta_P=\{\alpha\in\Delta\,|\,\mathfrak{g}_\alpha\subset\mathfrak{p}_C\}$. Here $\mathfrak{k}_C$ and $\mathfrak{p}_C$ are the respective complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ in $\mathfrak{g}_C$. Then $\Delta_K=\{\alpha\in\Delta\,|\,\alpha(H)=0\}$.

Let $\mathfrak{h}_R=i\mathfrak{h}_*=\{h\in\mathfrak{h}\,|\,\alpha(h)\in\boldsymbol{R}$ for all $\alpha\in\Delta\}$. Let $H=H_1, H_2, \cdots, H_l$ be a basis of $\mathfrak{h}_R$. Order $\mathfrak{h}^*_R\supset\Delta$ lexicographically relative to the ordered basis $H_1,\cdots,$ H_l. Let Δ^+ be the corresponding system of positive roots for Δ. Set $\Delta^+_K=\Delta^+\cap\Delta_K$, $\Delta^+_P=\Delta^+\cap\Delta_P$. Set $\mathfrak{p}^+=\sum_{\alpha\in\Delta^+_P}\mathfrak{g}_\alpha$. $\mathfrak{p}^-=\sum_{\alpha\in\Delta^+_P}\mathfrak{g}_{-\alpha}$. Then $\mathfrak{p}_C=\mathfrak{p}^+\oplus\mathfrak{p}^-$ and $\mathrm{ad}\,H|_{\mathfrak{p}^+}=I$, $\mathrm{ad}\,H|_{\mathfrak{p}^-}=-I$. Hence $\mathrm{ad}(\mathfrak{k}_C)\cdot\mathfrak{p}^\pm\subset\mathfrak{p}^\pm$. Also set $\mathfrak{n}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{n}^-=\sum_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha}$.

DEFINITION 3.1. A representation (π, V) of $\mathfrak{g}^C$ is said to *have highest weight* $\Lambda\in\mathfrak{h}^*=\mathfrak{h}_R^*\otimes \boldsymbol{C}$ if there is $v_0\in V$, $v_0\neq 0$ so that

(1) $\pi(H)v_0=\Lambda(H)v_0$ for $H\in\mathfrak{h}$,

(2) $\pi(\mathfrak{n}^+)v_0=0$,

(3) $\pi(U(\mathfrak{g}_C))v_0=V$.

Here $U(\mathfrak{g}_C)$ is the universal enveloping algebra of $\mathfrak{g}_C$.

DEFINITION 3.2. A representation (π, V) of $\mathfrak{g}_C$ is said to be $\mathfrak{g}$-*infinitesimally unitary* if there exists a positive definite inner product $\langle\ ,\ \rangle$ on V such that $\langle\pi(X)v, w\rangle+\langle v, \pi(X)w\rangle=0$ for all $X\in\mathfrak{g}$, $v, w\in V$.

Theorem 3.3. (Harish-Chandra [2]). *If (π, V) is a $\mathfrak{g}$-infinitesimally unitary representation of $\mathfrak{g}_C$ with highest weight Λ and if $\bar{V}$ is the Hilbert space completion of $(V, \langle\ \rangle)$ then there exists an irreducible unitary representation σ of G on $\bar{V}$ so that the differential of σ restricted to V is π.*

Proof. The only difference between the above statement and Theorem 4 of Harish-Chandra [2] is the irreducibility statement. Let $v_0\in V$ be as in Definition 3.1. It is easy to see that if $v\in V$ and $\pi(h)v=\Lambda(h)v$ for all $h\in\mathfrak{h}$ then $v=cv_0$ for some scalar c.
Suppose that $W\subset V$ is an invariant subspace. Then $W^\perp$ is invariant. Clearly $v_0\in W$ or $v_0\in W^\perp$. Hence $W=V$ or $W^\perp=V$. Thus π is indeed irreducible if it is $\mathfrak{g}$-infinitesimally unitary.

Lemma 3.4. *Let β be the largest root of Δ. If $\Lambda\in\mathfrak{h}^*$ and $\langle\Lambda, \beta\rangle=0$ ($\langle\ ,\ \rangle$ is the dual in bilinear form on $\mathfrak{h}^*$ to B on $\mathfrak{h}$) and if $\Lambda\neq 0$ then no representation with highest weight Λ can be $\mathfrak{g}$-infinitesimally unitary.*

Proof. Let $\mathfrak{g}_u=\mathfrak{k}\oplus i\mathfrak{p}$ and let $E_\alpha\in\mathfrak{g}_\alpha$ define a Weyl basis of $\mathfrak{g}_C/\mathfrak{h}$ relative to $\mathfrak{g}_u$. That is if τ is the conjugation of $\mathfrak{g}_C$ relative to $\mathfrak{g}_u(\tau(X+iY)=X-iY$, $X, Y\in\mathfrak{g}_u)$ then $\tau E_\alpha=-E_{-\alpha}$ and $[E_\alpha, E_{-\alpha}]=H_\alpha$ with $B(H_\alpha, H)=\alpha(H)$ for $H\in\mathfrak{h}$. If (π, V) is $\mathfrak{g}$-infinitesimally unitary relative to $\langle\ ,\ \rangle$ then $\langle\pi(X)v, w\rangle=-\langle v, \pi(\sigma(X))w\rangle$ for $X\in\mathfrak{g}_C$, $v, w\in V$. Here σ is conjugation of $\mathfrak{g}_C$ relative to $\mathfrak{g}$. Also $\sigma E_\alpha=-E_{-\alpha}$ for $\alpha\in\Delta_K$, $\sigma E_\alpha=E_{-\alpha}$ for $\alpha\in\Delta_P$.

Suppose now (π, V) is $\mathfrak{g}$-infinitesimally unitary with highest weight Λ and $\langle\Lambda, \beta\rangle=0$. Let v_0 be as in Definition 3.1 and assume that $\langle v_0, v_0\rangle=1$.

$$(1)\quad \pi(E_{-\beta})v_0=0.$$

In fact, since $\beta\in\Delta_P^+$ we see $\langle\pi(E_{-\beta})v_0, \pi(E_{-\beta})v_0\rangle=-\langle\pi(\sigma E_{-\beta})\pi(E_{-\beta})v_0, v_0\rangle$ $=-\langle\pi(E_\beta)\pi(E_{-\beta})v_0, v_0\rangle=-\langle\pi(H_\beta)v_0, v_0\rangle=-\Lambda(H_\beta)=-\langle\Lambda, \beta\rangle=0$ (since $\pi(E_\beta)$ $v_0=0$).

Now $-\beta$ is the lowest weight of (ad, $\mathfrak{g}_C$). Hence $\mathrm{ad}(U(\mathfrak{n}^+))E_{-\beta}=\mathfrak{g}_C$. ($U(\mathfrak{n}^+)$ the universal enveloping algebra of $\mathfrak{n}^+$). On the other hand $0=\pi(\mathfrak{n}^+U(\mathfrak{n}^+))\pi(E_{-\beta})v_0=\pi(\mathrm{ad}(\mathfrak{n}^+U(\mathfrak{n}^+))\cdot E_{-\beta})v_0$. But $\mathfrak{g}_C=\boldsymbol{C}E_{-\beta}\oplus\mathrm{ad}(\mathfrak{n}^+U(\mathfrak{n}^+))E_{-\beta}$. Hence

$\pi(\mathfrak{g}_C)v_0=0$. But then $\Lambda=0$ since $0=\pi(h)v_0=\Lambda(h)v_0$ for $H\in\mathfrak{h}$. Q.E.D.

The following lemma is more or less well known (Kostant [9]). We include a proof for the sake of completeness.

Lemma 3.5. *Let $W^1=\{s\in W_C \mid s\Delta^+\supset\Delta_K^+\}$. Here W_C is the Weyl group of the pair $(\mathfrak{g}_C, \mathfrak{h})$. Let $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. If $s\in W_C$ define $l(s)$ to be the order of $s\Delta^+\cap\Delta^-$ $(\Delta^-=\{-\alpha\mid\alpha\in\Delta^+\})$. Let σ_k denote the representation of $\mathfrak{k}_C$ on $\wedge^k\mathfrak{p}^-$. Let for $\lambda\in\mathfrak{h}_R^*$, Δ_K^+-dominant integral (that is $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is a nonnegative integer for $\alpha\in\Delta_K^+$), τ_λ denote the irreducible finite dimensional representation of $\mathfrak{k}_C$ with highest weight* λ (c.f. Wallach [18], chapter 4).

(1) *If $s\in W_C$ then $s\rho-\rho$ is Δ_K^+-dominant integral if and only if $s\in W^1$.*

(2) $\sigma_k=\sum_{\substack{s\in W^1,\\ l(s)=k}}\tau_{s\rho-\rho}$.

Proof. (1) If $s\in W_C$ and $\langle s\rho-\rho,\alpha\rangle\geq 0$ for all $\alpha\in\Delta_K^+$ then $\langle s\rho,\alpha\rangle>0$ for all $\alpha\in\Delta_K^+$. Thus $\alpha\in s\Delta^+$ for all $\alpha\in\Delta_K^+$. Let α be a simple root in Δ_K^+ (this is equivalent to $2\langle\rho,\alpha\rangle=\langle\alpha,\alpha\rangle$). Then if $s\in W^1$ then $\langle s\rho-\rho,\alpha\rangle=\langle\rho-s^{-1}\rho, s^{-1}\alpha\rangle=\langle\rho, s^{-1}\alpha\rangle-\langle\rho,\alpha\rangle=\langle\rho, s^{-1}\alpha\rangle-\frac{\langle\alpha,\alpha\rangle}{2}$. But $\Delta_K^+\subset s\Delta^+$ thus $s^{-1}\Delta_K^+\subset\Delta^+$. Thus, since $\langle\rho,\gamma\rangle\geq\langle\gamma,\gamma\rangle/2$ for all and $\gamma\in\Delta^+$ and $\langle s^{-1}\alpha, s^{-1}\alpha\rangle=\langle\alpha,\alpha\rangle$, we see that $\langle s\rho-\rho,\alpha\rangle\geq 0$ for $\alpha\in\Delta_K^+$, α simple. But every $\alpha\in\Delta_K^+$ is a non-negative integral combination of simple roots in Δ_K^+. This proves (1).

If $h\in\mathfrak{h}$ then $\operatorname{tr}\sigma_k(\exp h)=\sum_{\substack{\beta_1,\cdots,\beta_k\in\Delta_P^+,\\ \text{distinct}}} e^{-(\Sigma\beta_i(h))}$.

Thus if $\chi_+(h)=\sum\operatorname{tr}\sigma_{2k}(\exp h)$ and $\chi_-(h)=\sum\operatorname{tr}\sigma_{2k+1}(\exp(h))$ then $\chi_+(h)-\chi_-(h)=\prod_{\alpha\in\Delta_P^+}(1-e^{-\alpha(h)})$. Hence $\chi_+(h)-\chi_-(h)=\prod_{\Delta\in\Delta^+}(1-e^{-\alpha(h)})/\prod_{\alpha\in\Delta_K^+}(1-e^{-\alpha(h)})$. But $e^{\rho(h)}\prod_{\alpha\in\Delta^+}(1-e^{-\alpha(h)})=\sum_{s\in W_C}\det(s)e^{s\rho(h)}$, $e^{\rho_K(h)}\prod_{\alpha\in\Delta_K^+}(1-e^{-\alpha(h)})=\sum_{s\in W_K}\det(s)e^{s\rho_K(h)}$ (here $\rho_K=\frac{1}{2}\sum_{\alpha\in\Delta_K^+}\alpha$) (see for example Wallach [18], Lemma 4.9.5.). Hence,

$$\chi_+(h)-\chi_-(h)=\Big(\sum_{s\in W_C}\det(s)e^{s\rho-\rho+\rho_K}\Big)\Big/\sum_{s\in W_K}\det(s)e^{s\rho_K}.$$

Now $\sigma_k(\exp tH)=e^{kt}I$. Thus the representations $\sum\oplus\sigma_{2k}$ and $\sum\oplus\sigma_{2k+1}$ of $\mathfrak{p}_C$ are disjoint. Also if $\sigma_k=\sum n_k(\lambda)\tau_\lambda$, then the Weyl character formula (c.f. Wallach [18], 4.9.6.) says $\big(\sum_{s\in W_K}\det(s)e^{s\rho_K}\big)(\chi_+-\chi_-)=\sum_k(-1)^k\sum n_k(\lambda)\sum_{s\in W_K}\det(s)e^{s(\lambda+\rho_K)}$.

Thus we see

$$\sum_k(-1)^k\sum_\lambda n_k(\lambda)\sum_{s\in W_K}\det(s)e^{s(\lambda+\rho_K)}=\sum_{s\in W_C}\det(s)e^{s\rho-\rho+\rho_K}.$$

From this, the above observations and (1), we see that $n_k(\lambda)=0$ or 1 and all of the highest weights appearing with $n_k(\lambda)\neq 0$ must be of the form $s\rho-\rho$, $s\in$

W^1. Now, if $s\in W_C$, $s\rho-\rho=\sum\gamma$ the sum taken over the elements of $s\Delta^+\cap\Delta^-$. If $s\in W^1$ then $s\Delta^+\cap\Delta^-\subset\Delta_P^-=\{-\alpha\mid\alpha\in\Delta_P^+\}$.

(a) If $s\in W^1$ and if $\alpha\in\Delta_K^+$, $\gamma\in s\Delta^+\cap\Delta^-$ then if $\alpha+\gamma\in\Delta$, $\alpha+\gamma\in s\Delta^+\cap\Delta^-$.

Indeed, if $\alpha\in\Delta_K^+$ and $\gamma\in s\Delta^+\cap\Delta^-$ and if $\alpha+\gamma\in\Delta$ then since $\alpha=s\delta$, $\delta\in\Delta^+$ and $\gamma=s\mu$, $\mu\in\Delta^+$, $\alpha+\gamma=s(\delta+\mu)$, $\delta+\mu\in\Delta^+$. But $\gamma\in\Delta_P^-$ and if $\alpha\in\Delta_K$, $\alpha+\gamma\in\Delta$ then $\alpha+\gamma\in\Delta_P^-$. Thus $\alpha+\gamma\in\Delta^-\cap s\Delta^+$. This proves ($a$).

But now (a) implies that if $\alpha\in\Delta_K^+$ then $\sigma_k(E_\alpha)\cdot(E_{-\beta_1}\wedge\cdots\wedge E_{-\beta_k})=0$ for $s\Delta^+\cap\Delta^-=\{-\beta_1,\cdots,-\beta_k\}$. Thus $n_k(s\rho-\rho)\neq0$ for $s\in W^1$. The lemma is now completely proven.

Lemma 3.6. *If G has split rank$>k$ and if $s\in W^1$ with $l(s)\leq k$ then $\langle s\rho-\rho,\beta\rangle=0$ (β the largest root).*

Proof. Let $\alpha_1,\cdots,\alpha_l$ be the simple roots in Δ^+. We may assume $\alpha_1\in\Delta_P^+$, $\alpha_j\in\Delta_K^+$, $j\geq2$. Let s_j be the Weyl reflection about the hyperplane $\alpha_j=0$ $\left(s_j\lambda=\lambda-\dfrac{2\langle\lambda,\alpha_j\rangle\alpha_j}{\langle\alpha_j,\alpha_j\rangle}\right)$.

Let $\gamma_1,\cdots,\gamma_r$ be elments of Δ_P^+ so that

(a) $\gamma_1=\alpha_1$

(b) if $i\neq j$, $\gamma_i\pm\gamma_j\notin\Delta$

(c) r is maximal subject to (a), (b).

Then $r=$split rank (G) (See Harish-Chandra [3]).

(d) Let $\mathfrak{h}^-=\sum\boldsymbol{C}H\gamma_i$. Then if $\alpha\in\Delta_K^+$, $\alpha|\mathfrak{h}^-$ is of one of the following forms: $\frac{1}{2}(\gamma_i-\gamma_j)$, $i>j$, $-\frac{1}{2}\gamma_j$, 0. If $\alpha\in\Delta_P^+$ then $\alpha|\mathfrak{h}^-$ is of one of the following forms $\frac{1}{2}(\gamma_j+\gamma_j)$, $i\leq j$ or γ_j, $\frac{1}{2}\gamma_j$ (see Harish-Chandra [3]).

We prove by induction on k:

(i) If $\{\alpha_1, s_1\alpha_{i_2}, s_1s_{i_2}\alpha_{i_3},\cdots,s_1s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k}\}\subset\Delta_P^+$ and are distinct and if $\beta_j=s_1s_{i_2}\cdots s_{i_{j-1}}\alpha_{i_j}$ then there are at most k of the γ_i so that $\langle\sum\beta_i,\gamma_i\rangle>0$.

(1) $k=1$ is clear since then $\sum\beta_i=\alpha_1$ and $\langle\gamma_i,\gamma_j\rangle=0$ if $i\neq j$.

(2) Suppose that the result is true for k. Set $\beta_{k+1}=s_{i_1}\cdots s_{i_k}\alpha_{i_{k+1}}\in\Delta_P^+$, $\beta_{k+1}\neq\beta_j$, $j\leq k$. Let $\gamma_{j_1},\cdots,\gamma_{j_u}$ be such that $\langle\gamma_{j_i},\sum_{j\leq k}\beta_j\rangle>0$. Then $u\leq k$. Now relabel the γ_i's so that $\gamma_1=\alpha_1$ (as before) $\gamma_{j_i}=\gamma_i$, $i\leq u$. (d) above implies $\beta_j|\mathfrak{h}^-\in\sum_{i\leq u}\boldsymbol{R}\gamma_i$ for $j\leq k$.

(I) $\beta_{k+1}|\mathfrak{h}^-=\frac{1}{2}(\gamma_i+\gamma_j)$, $i<j$.

If $u<k$ then $\{\gamma_1,\cdots,\gamma_u,\gamma_i,\gamma_j\}$ has at most $k+1$ elements and the induction is complete. If $u=k$ then in order for the lemma to be false we must have $\sum_{j=1}^k\beta_j|\mathfrak{h}^-=\sum_{j=1}^k c_j\gamma_j$, $c_j>0$ and $\beta_{k+1}|\mathfrak{h}^-=\frac{1}{2}(\gamma_{k+1}+\gamma_{k+2})$. But now $\beta_{k+1}=s_{i_1}\cdots s_{i_k}\alpha_{i_{k+1}}$. Thus $\beta_{k+1}=s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_{k+1}}+d_k\alpha_{i_k})=d_k\beta_k+s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_{k+1}}=\sum_{j=1}^k d_j\beta_j+$

$\alpha_{i_{k+1}}$. But then $\alpha_{i_{k+1}}|\mathfrak{h}=\frac{1}{2}(\gamma_{k+1}+\gamma_{k+2})$. But this is impossible since if $i_{k+1}=1$, $\alpha_{i_{k+1}}|\mathfrak{h}^-=\gamma_1$ if $i_{k+1}\geq 2$, $\alpha_{i_{k+1}}|\mathfrak{h}^-=-\frac{1}{2}\gamma_t$ or $\frac{1}{2}(\gamma_t-\gamma_s)$, $s<t$. Thus in case (I) the induction is complete.

The only other possibility is $\beta_{k+1}|\mathfrak{h}=\gamma_t$ or $\frac{1}{2}\gamma_t$ for some $1\leq t\leq r$. But then the set $\{\gamma_1, \cdots, \gamma_u, \gamma_t\}$ has at most $k+1$ elements. (i) is now completely proved.

We now prove the lemma. Suppose $s\in W^1$ and $\langle s\rho-\rho, \beta\rangle\neq 0$. Since $s\rho-\rho$ is a sum of netgative roots and β is the highst weight of the adjoint representation $\langle s\rho-\rho, \beta\rangle<0$. If $\gamma\in\Delta_P^+$ then again since β is the highest weight of $\mathfrak{p}$ as a representation of $\mathfrak{k}_C$, $\gamma=\beta-\sum_{i=2}^{l} n_i\alpha_i$, $n_i\geq 0$. Hence $\langle s\rho-\rho, \gamma\rangle=\langle s\rho-\rho, \beta\rangle-\sum n_i\langle s\rho-\rho, \alpha_i\rangle<0$ since $\langle s\rho-\rho, \alpha_i\rangle\geq 0$, $i\geq 2$ by Lemma 3.5.

Hence if $s\in W^1$ and $\langle s\rho-\rho, \beta\rangle\neq 0$ then $\langle\rho-s\rho, \gamma\rangle>0$ for all $\gamma\in\Delta_P^+$. If $s\in W^1$, $l(s)=k$ then $s=s_{i_1}\cdots s_{i_k}$ and $s\Delta^+\cap\Delta^-=\{-\alpha_{i_1}, -s_{i_1}\alpha_{i_2}, \cdots, -s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_k}\}\subset\Delta_P^-$ and contains precisely k elements. (c.f. Wallach [18], 8.9.13). But then $\alpha_{i_1}\in\Delta_P^+$ hence $i_1=1$. If $\beta_j=s_1 s_{i_2}\cdots s_{i_{j-1}}\alpha_{i_j}$ then $\rho-s\rho=\sum_{j\leq k}\beta_j$. If $\langle\rho-s\rho, \beta\rangle\neq 0$ then by the above $\langle\rho-s\rho, \gamma_j\rangle>0$ for $j=1, \cdots, r$. But then (i) implies $r\leq k$. Q.E.D.

The following lemma is due to Parthasarathy [15]. We include the proof since it is only implicitely stated in [15].

Lemma 3.7. *Let V_λ be the irreducible (finite dimensional) representation of $\mathfrak{k}_C$ with highest weight λ. Let $\Lambda^p\mathfrak{p}^-\otimes V_\lambda=\sum n_p(\lambda:\mu)V_\mu$ as a representation of $\mathfrak{k}_C$. Suppose that λ is Δ^+-dominant integral. If π is an irreducible unitary representation of G so that $\pi(\Omega)=|\lambda+\rho|^2-|\rho|^2=C_\lambda$ and $\sum_{n_p(\lambda,:\mu)\neq 0}(\pi|_K: V_\mu)\neq 0$, then π is infinitesimally equivalent with the irreducible representation with highest weight μ for some μ with $n_p(\lambda:\mu)\neq 0$.*

Proof. We retain the notation of Lemma 3.4. Let $H_1, \cdots, H_l$ be a basis of $\mathfrak{h}$ so that $B(H_i, H_j)=\delta_{ij}$.
Then $\Omega=\sum_{\alpha\in\Delta}E_\alpha E_{-\alpha}+\sum_{i=1}^{l}H_i^2$. Set $\Omega_K=\sum_{\alpha\in\Delta_K}E_\alpha E_{-\alpha}+\sum_{i=1}^{l}H_i^2$. Set $\Omega_p=\sum_{\alpha\in\Delta_P}E_\alpha E_{-\alpha}$.

Let (π, H) be an irreducible unitary representation of G so that $\pi(\Omega)=C_\lambda$ and $(\pi|_K: V_\mu)\neq 0$ for some μ so that $n_p(\lambda:\mu)\neq 0$. Let $v_\mu\in H$ a unit vector such that

(1) $\pi(E_\alpha)v_\mu=0$, $\alpha\in\Delta_K^+$

(2) $(U(\mathfrak{k}_C))v_\mu$ is equivalent with V_μ as a representation of $\mathfrak{k}_C$.

Then $\pi(\Omega_K)v_\mu=(\langle\mu+\rho_K, \mu+\rho_K\rangle-\langle\rho_K, \rho_K\rangle)v_\mu$ (c.f. Wallach [18], 5.6.4.). Hence $\sum_{\alpha\in\Delta_p}\pi(E_\alpha E_{-\alpha})v_\mu=(C_\lambda-|\mu+\rho_K|^2+|\rho_K|^2)v_\mu$. But $\sum_{\alpha\in\Delta_P}\pi(E_\alpha E_{-\alpha})v_\mu=2\sum_{\alpha\in\Delta_P^+}\pi(E_{-\alpha})\pi(E_\alpha)v_\mu+\sum_{\alpha\in\Delta_P^+}\pi(H_\alpha)v_\mu=2\sum_{\alpha\in\Delta_P^+}\pi(E_{-\alpha}E_\alpha)v_\mu+2\sum_{\alpha\in\Delta_P^+}\mu(H_\alpha)$

v_μ. We therefore see (3) $2\sum_{\alpha\in\Delta_P^+}\pi(E_{-\alpha})\pi(E_\alpha)v_\mu=(C_\lambda-\langle\mu+2\rho,\mu\rangle)v_\mu=(|\lambda+\rho|^2-|\mu+\rho|^2)v_\mu$.

Now if $\alpha\in\Delta_P$ then $\langle\pi(E_\alpha)v, w\rangle=-\langle v, \pi(E_{-\alpha})w\rangle$, $v, w\in$ domain $\{\pi(E_\alpha), \pi(E_{-\alpha})\}$. Using this we see that if we take the inner product of the left hand side of (3) with v_μ we have

(4) $-2\sum_{\alpha\in\Delta_P^+}\|\pi(E_\alpha)v_\mu\|^2=(|\lambda+\rho|^2-|\mu+\rho|^2)$.

But every highest weight of $V_\lambda\otimes\Lambda^p\mathfrak{p}^-$ is of the form $\lambda-\langle Q\rangle$ where $Q\subset\Delta_P^+$ is a subset and $\langle Q\rangle=\sum_{\alpha\in Q}\alpha$. But if $Q\subset\Delta^+$, $|\lambda+\rho|^2-|\lambda-\langle Q\rangle+\rho|^2\geq 0$. (See Kostant [9]). Hence $\pi(E_\alpha)v_\mu=0$ for $\alpha\in\Delta_P^+$. This clearly implies π has highest weight μ. Q.E.D.

NOTE: Case (1) of Proposition 2.2 follows from Lemma 3.7 using $\lambda=0$ since $\mathfrak{p}^-$ is irreducible and thus there is only one possible μ with $n_1(0:\mu)\neq 0$.

Corollary 3.8. *Suppose that* split rank $G>k$. *If* $k>0$ *and if* $\Lambda^k\mathfrak{p}^-=\sum n_k(\mu)V_\mu$ *then there does not exist a unitary irreducible representation* (π, H) *of* G *so that*

(1) $\pi(\Omega)=0$

(2) $\sum n_k(\mu)\,[\pi|_K: V_\mu]\neq 0$.

Proof. If such a (π, H) existed then it would have to have highest weight μ for some μ with $n_k(\mu)\neq 0$. But then $\mu=s\rho-\rho$ for some $s\in W^1$, $l(s)=k$. But then $\langle s\rho-\rho, \beta\rangle\neq 0$ by Lemma 3.4 and Lemma 3.6 yields a contradiction.

NOTE. The above result is best possible in light of Lemma 2.1. Since split rank $SU(n,1)=1$ and π_1 in the notation of Lemma 2.1 satisfies (1), (2) above with $k=1$.

4. The Betti numbers of $\Gamma\backslash X$

Let G be a connected, simple Lie group with finite center and let K be a maximal compact subgroup of G. Let $\Gamma\subset G$ be a discrete subgroup of G so that Γ has no elements of finite order and $\Gamma\backslash G$ is compact. Let $X=G/K$. Then $\Gamma\backslash X$ is a compact locally symmetric space. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to K. Let dg be a Haar measure on G and let $d\dot{g}$ denote the corresponding G invariant measure on $\Gamma\backslash G$. Let π_Γ be the regular representation of G on $L^2(\Gamma\backslash G)$. Then it is well known that as a representation of G, $\pi_\Gamma=\sum_{\omega\in\hat{G}}N_\Gamma(\omega)\omega$ (here $\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G) and $N_\Gamma(\omega)<\infty$.

Let $(\Lambda^p(\mathrm{Ad}|_K), \Lambda^p\mathfrak{p}_C)=\sum m_{p,\lambda}\tau_\lambda$ with τ_λ the irreducible unitary representation of K with highest weight λ. If $[\mathfrak{k},\mathfrak{k}]\neq\mathfrak{k}$ then $\mathfrak{p}_C=\mathfrak{p}^+\oplus\mathfrak{p}^-$ as a representation of K and let

$$(\Lambda^p\mathrm{Ad}|_K\otimes\Lambda^q\mathrm{Ad}|_K, \Lambda^p\mathfrak{p}^+\otimes\Lambda^q\mathfrak{p}^-)=\sum n_{p,q;\lambda}\tau_\lambda\,,$$

τ_λ the irreducible representation of K with highest weight λ.

Theorem 4.1. (Matsushima [13], Matsushima-Murakami [14]).

(1) $b_p(\Gamma\backslash X)=\sum_{\pi\in\hat{G}_0}N_\Gamma(\pi)\sum_{\lambda,m_{p,\lambda}\neq 0}(\pi|_K:\tau_\lambda)$. (2) *If* $[\mathfrak{k},\mathfrak{k}]\neq\mathfrak{k}$ *then* $\Gamma\backslash X$ *is a Kähler manifold and* $b_{p,q}(\Gamma\backslash X)=\sum_{\pi\in\hat{G}_0}N_\Gamma(\pi)\sum_{\lambda,n_{p,q;\lambda}\neq 0}(\pi|_K:\tau_\lambda)$. *Here* $\hat{G}_0=\{\pi\in\hat{G}\mid\pi(\Omega)=0\}$.

(2) above combined with Corollary 3.8 immediately implies

Theorem 4.2. *If* rank $X=k$ *then* $b_{0,q}(\Gamma\backslash X)=0$ *for* $0<q<k$.

Corollary 4.3. (Matsushima [12]). *If X is irreducible and Hermitian symmetric of rank greater than* 1 *then* $b_1(\Gamma\backslash X)=0$.

Proof. $b_1(\Gamma\backslash X)=b_{0,1}(\Gamma\backslash X)+b_{1,0}(\Gamma/X)$. But $b_{1,0}(\Gamma\backslash X)=b_{0,1}(\Gamma\backslash X)$ since $\Gamma\backslash X$ is Kähler.

We also note (using the notation of §2)

Lemma 4.4. *If G is locally isomorphic with $SU(n,1)$ then* $b_1(\Gamma\backslash X)=2N_\Gamma(\pi_1)$. *If G is locally isomorphic with $SO_e(n,1)$ and $n\geq 3$ then* $b_1(\Gamma\backslash X)=N_\Gamma(\pi_1)$. *Otherwise if G is of split rank* 1 *then* $b_1(\Gamma\backslash X)\geq N_\Gamma(\pi_1)$.

We conclude this paper by looking at the example $G=SU(2,1)$. Let us denote, π_1, by $\pi^{0,1}$. Let Λ be the highest weight of $\Lambda^2\mathfrak{p}^-$. Then the irreducible representation of G with highest weight Λ (in the sense of §3) is a holomorphic discrete series representation. It is not integrable however so Langlands' formula [11] does not apply. (It can also be shown that the formula of Hotta-Parthasarathy [6] does not apply either). Denote this representation by $\pi^{0,2}$. Let $\pi^{1,1}$ be as in Proposition 2.3. Then $b_{0,1}(\Gamma\backslash X)=N_\Gamma(\pi^{0,1})$, $b_{0,2}(\Gamma\backslash X)=N_\Gamma(\pi^{0,2})$, $b_{1,1}(\Gamma\backslash X)=N_\Gamma(\pi^{1,1})+1$. If we normalize dg so that vol $(\Gamma\backslash G)=\int_{\Gamma/G}d\dot{g}=\chi(\Gamma\backslash X)$ the Euler number of $\Gamma\backslash X$ then the Max Noether formula combined with the Hirzebruch proportionality principle implies

(*a*) $1-b_{0,1}(\Gamma\backslash X)+b_{0,2}(\Gamma\backslash/X)=\frac{1}{3}\text{vol}(\Gamma\backslash G)$.

also $\chi(\Gamma\backslash X)=\sum_{p=0}^{4}(-1)^p b_p(\Gamma\backslash X)$.

Poincaré duality implies

(*b*) $2-2b_1(\Gamma\backslash X)+b_2(\Gamma\backslash X)=\text{vol}(\Gamma\backslash G)$.

Hence we have

(*a'*) $1-N_\Gamma(\pi^{0,1})+N_\Gamma(\pi^{0,2})=\frac{1}{3}\text{vol}(\Gamma\backslash G)$

(*b'*) $3-4N_\Gamma(\pi^{0,1})+2N_\Gamma(\pi^{0,2})+N_\Gamma(\pi^{1,1})=\text{vol}(\Gamma\backslash G)$.

Now since $\text{vol}(\Gamma\backslash G)>0$ we see that if $b_{0,2}(\Gamma\backslash X)=0$ then $b_{0,1}(\Gamma\backslash X)=0$ hence $\text{vol}(\Gamma\backslash G)=3$. Hence $b_2(\Gamma\backslash X)=1$. Thus (as is well known in this case) if Γ is such that $N_\Gamma(\pi_{0,2})=0$ then the real cohomology ring of $\Gamma\backslash X$ is generated by the

Kähler class and is isomorphic with the real cohomology ring of $\boldsymbol{C}P^2$.

We also note that (a'), (b') combined imply $N_\Gamma(\pi^{1,1})=N_\Gamma(\pi^{0,2})+N_\Gamma(\pi^{0,1})$. We can see no reason in the harmonic analysis of $L^2(\Gamma\backslash G)$ why this should be true. Finally we note that in (a'), $\pi^{0,2}$ is discrete series but $\pi^{0,1}$ is a so called "trash representation". That is, it is non-tempered but its character has support in the elliptic regular elements. Paul Sally and the authors have named these representations *trash* because they seem to be the reason why the expected formulae for the multiplicities of discrete series are not right. That is, if the trash has been disposed of the formula is correct. Note that the trivial representation is a trash representation.

Similar coupling of non-integrable discrete series and trash representations have been found by Paul Sally and the second author of this paper for the two-fold cover of $SL(2, \boldsymbol{R})$.

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