Lie Algebra Cohomology and Holomorphic Continuation of Generalized Jacquet Integrals

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Introduction

In this paper there are two types of theorems. The first are generalizations of vanishing theorems of Kostant [K] and Lynch [L]. The second are the holomorphic continuation of certain integrals. The proofs of the two seemingly unrelated types of results have in common the use of certain operators, $Q$, which are non-commutative analogues of the standard Euler operator on the space of polynomials in several variables.

We now describe an important class of examples of the results. Let $G$ be a reductive group of inner type. Let $g$ denote the Lie algebra of $G$ and let $Y$ be a nilpotent element in $g$. Then ([Ja, p. 99, Lemma 8]) there exist elements $X, H \in g$ such that $[X, Y] = H$ and $[H, X] = 2X, [H, Y] = -2Y$. Fix a Cartan involution such that $\theta H = -H$. Let $\mathfrak{u}$ be the Lie subalgebra of $g$ generated by $\mathfrak{g}_\mathfrak{u} := \{ x \in g | [H, x] = 2x \}$. Let $z \in \mathbb{C}[\theta^{-1}]$ and let $\hat{\psi}(x) := zB(Y, x)$ for $x \in \mathfrak{u}$. If $V$ is a $g$-module then we define a new action $\hat{\pi}_z$ of $\mathfrak{u}$ on $V$ by $\hat{\pi}_z(x)v := xv - \hat{\psi}(x)v$. We denote this $\mathfrak{u}$-module by $V \otimes \mathbb{C}_z$. The main theorem on Lie algebra cohomology implies

**Theorem.** Let $V$ be a $g$-module such that if $v \in V$ then $\pi_\alpha(x)^k v = 0$ for all $x \in \mathfrak{u}$ for some $k = k(v)$. Then $H^i(\mathfrak{u}, V \otimes \mathbb{C}_z) = 0$ for $i > 0$.

We now describe the other type of results. Let $\mathfrak{g}$ be the sum of the eigenspaces for $ad H$ with non-negative eigenvalue. Let $P = \{ g \in G | Ad(g) \mathfrak{g} \subset \mathfrak{g} \}$ and put $M = P \cap \theta(P)$. Let $(\alpha, H_\alpha)$ be a finite dimensional irreducible representation of $M$. Put $g = \{ Z \in \mathfrak{m} | [Z, \mathfrak{m}] = 0, \theta Z = -Z \}$. If $\nu \in g_\mathfrak{z}$ then let $(\pi_\nu, \rho_\nu, J_\nu)$ be the corresponding (degenerate) principal series representation (see § 6). Let $\tau = \exp (\pi/2(\hat{\lambda} - Y))$.

The analytic results involve the study of integrals of the form

$$J_{\tau, \alpha, \nu}(f) = \int_{\mathfrak{u}} e^{iB(Y, u)} f_{\tau, \alpha}(\tau \exp (u)) du. \quad (*)$$

Received December 26, 1986.

\(^{22}\) Research partially supported by a grant from the National Science Foundation.
(See § 6 for unexplained notation.) If \( Y \) is a principal nilpotent element of \( \mathfrak{g} \) then \( P \) is a minimal parabolic subalgebra and \( \mathfrak{u} \) is the nilradical of \( \mathfrak{p} \). Also \( \gamma(\exp(x)) = e^{iB(Y, x)} \) defines the most general "generic character" of \( N \). In this case our results imply that \( J_{\mathfrak{p}, \mathfrak{v}} \) has a weakly holomorphically (not just meromorphic) continuation in \( \mathfrak{v} \) to \( \mathfrak{a}_0^\mathfrak{s} \) as an operator on the \( C^\infty \) vectors. This result in the case when \( G \) has reduced rank 1 is due to Schiffman [S]. In the case when \( G \) is split or complex then the result (on \( K \)-finite vectors) is due to Jacquet [J] (thus the designation in the title). There are other papers with special cases of this theorem ([HI], [HIJ]).

If \( Y \) is not principal then the results are more difficult to describe. (See § 5, 6, 7). We will instead give an example. Let \( G = \text{Sp}(n, \mathbf{R}) \) which we realize as the group of all linear transformations of \( \mathbf{R}^{2n} \) that preserve the form \( \omega(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \). We take

\[
Y = \begin{bmatrix} 0 & 0 \\ I_{p \times q} & 0 \end{bmatrix},
\]

with

\[
I_{p \times q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}
\]

with \( I_j \) the \( j \times j \) identity matrix and \( p + q = n \). Then we can take

\[
H = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}
\]

with \( I \) the \( n \times n \) identity matrix and \( X = Y^T \). In this case

\[
\mathfrak{u} = \left\{ \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} : S = S^T, \ S \in \mathbf{R}^{n \times n} \right\}.
\]

The corresponding integrals are of the form

\[
\int e^{i \mathbf{tr}(i_{p \times q}) f_s, x} \left( J \exp \left( \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \right) \right) dS
\]

the integration over the space of all symmetric \( n \times n \) matrices, \( x \in \mathbf{R}\{-0\} \) and

\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

In this example we show that these integrals have holomorphic continuations for smooth \( f \). (For precise results see § 7). If \( G = SU(n, n) \),
SL(2n, \mathbb{R}), SO(2n, 2n) there are parabolic subgroups with similar descriptions to \( P \) above and our results imply the holomorphic continuation of the corresponding integrals (see §4 for a large class of examples). The proofs are given in such a way that if one can prove the conjecture at the end of section 5 then the most general integral (\( \ast \)) has a holomorphic continuation to all of \( \mathfrak{g}^e \).

As we indicated above the results rest on the algebraic properties of certain generalizations of Euler operators. Lemma 2.1 is the contains the key to these operators. Our proof of this Lemma involves the determination of the minimal polynomial of an element of the group algebra of the symmetric algebra on \( n \) letters (see the appendix).

The material of this paper is related to a long term joint project with Roberto Miatello. We thank Roe Goodman for his constructive criticism of the results of this paper as they evolved. We thank the Taniguchi Foundation for having sponsored the workshop and symposium in Kyoto during the summer of 1986.

§ 1. Some observations about unipotent representations

Let \( \mathfrak{n} \) be a Lie algebra over \( \mathbb{C} \). If \( M \) is an \( \mathfrak{n} \)-module and if \( m \in M \) then we set \( n^\mathfrak{n} m = m \) and if \( n^\mathfrak{n} m \) has been defined then \( n^\mathfrak{n} + 1 m = \{ XV | v \in n^\mathfrak{n} m \} \). Let \( \mathcal{N} \) denote the category of all \( \mathfrak{n} \)-modules \( M \), such that if \( m \in M \) then there exists \( k = k(m) \) such that \( n^k m = (0) \). If \( M \in \mathcal{N} \) and if \( M_n = \{ m \in M | n m = 0 \} = (0) \) then \( M = (0) \). Let \( M \in \mathcal{N} \). Set \( M^s = \{ m \in M | n^s m = (0) \} \). Then \( M^0 = (0) \subset M^1 \subset M^s \subset M^2 \subset \cdots \) and \( \bigcup M^j = M \).

If \( M \) is an \( \mathfrak{n} \)-module then let \( H^1(\mathfrak{n}, M) \) denote the usual Lie algebra cohomology space of \( \mathfrak{n} \) with coefficients in \( M \) (cf. [BW, Chapter I]). For our purposes the most important cohomology spaces are the zeroth which is just \( M^n \) and the first (which we now recall). Let \( Z^1(\mathfrak{n}, M) \) denote the space of all \( \omega : \mathfrak{n} \rightarrow M \) such that \( \omega([X, Y]) = X \omega(Y) - Y \omega(X) \) for all \( X, Y \in \mathfrak{n} \). Let \( B^1(\mathfrak{n}, M) \) denote the space of those \( \omega \) of the form \( \omega(X) = Xm \) with \( m \in M \) fixed. Then \( H^1(\mathfrak{n}, M) = Z^1(\mathfrak{n}, M)/B^1(\mathfrak{n}, M) \).

Lemma 1.1. Let \( M, V \in \mathcal{N} \). Assume that \( H^1(\mathfrak{n}, V) = (0) \). If \( A \in \text{Hom}_\mathfrak{n}(M^n, V^n) \) then there exists \( T \in \text{Hom}_\mathfrak{n}(M, V) \) such that \( T \) agrees with \( A \) on \( M^n \). If \( A \) is injective then so is \( T \). If \( A \) is bijective and if \( H^1(\mathfrak{n}, M) = (0) \) then \( T \) is surjective.

Proof. Define \( T \) to be \( A \) on \( M^j \). Suppose that \( T \) has been defined on \( M^j \) for \( j \geq 1 \) as an \( \mathfrak{n} \)-module homomorphism. If \( m \in M^{j+1} \) then \( Xm \in M^j \) for \( X \in \mathfrak{n} \). Set \( \omega_m(X) = T(Xm) \). Then it is easily verified that \( \omega_m \in Z^1(\mathfrak{n}, V) \). Hence there exists \( v \in V \) such that \( T(Xm) = Xv \) for all \( X \in \mathfrak{n} \). Let \( \{ m_a \} \) be a linearly independent set in \( M^{j+1} \) that defines a basis modulo
Let $v_\alpha \in V$ be such that $T(Xm_\alpha) = XV_\alpha$ for all $X \in \mathfrak{g}$. Define $Tm_\alpha = v_\alpha$ and extend $T$ by linearity to the span of the $v_\alpha$. This defines a linear map of $M^{j+1}$ into $V^{j+1}$. If $m \in M^{j+1}$ then $m = \sum a_\alpha m_\alpha + u$ with $u \in M^j$. Thus $T(Xm) = \sum a_\alpha T(Xm_\alpha) + T(Xu) = \sum a_\alpha XV_\alpha + XT(u) = \sum a_\alpha XT(m_\alpha) + XT(u) = XT(m)$. This implements the "construction" of $T$.

Assume that $A$ is injective. We show that $T$ is injective on $M^j$ by induction on $j$. If $j = 1$ then $T$ is injective on $M^1$ by hypothesis. So assume that $T$ is injective on $M^j$. Suppose that $m \in M^{j+1}$ and $Tm = 0$. If $X \in \mathfrak{g}$ then $T(Xm) = XTm = 0$. Since $Xm \in M^j$ this implies that $Xm = 0$ for all $X \in \mathfrak{g}$. Hence $m \in M^j$ so $m = 0$.

Suppose now that $H^j(\mathfrak{g}, M) = (0)$ and that $A$ is bijective. Then the short $\mathfrak{g}$-module exact sequence

$$0 \rightarrow M^s \xrightarrow{A} V^s \rightarrow (V/ TM)^s \rightarrow H^j(\mathfrak{g}, M) = (0).$$

induces the long(er) $\mathfrak{g}$-module exact sequence

$$0 \rightarrow M^s \xrightarrow{A} V^s \rightarrow (V/ TM)^s \rightarrow H^j(\mathfrak{g}, M) = (0).$$

Since $A$ is assumed to be bijective, this implies that $(V/ TM)^s = (0)$. Hence $V/ TM = (0)$. So $V = TM$. This completes the proof of the Lemma.

If $M$ is an $\mathfrak{g}$-module then put $M[\mathfrak{n}] = \{m \in M | \mathfrak{n}^k M = (0) \text{ for some } k\}$. If $W$ is a complex vector space then we put an $\mathfrak{g}$-module structure on $\text{Hom}_C(U(\mathfrak{n}), W)$ by setting $Xf(n) = f(nx)$ or $X \in \mathfrak{g}$, $n \in U(\mathfrak{n})$ and $f \in \text{Hom}_C(U(\mathfrak{n}), W)$. Set $N(\mathfrak{n}) = \text{Hom}_C(U(\mathfrak{n}), W)[\mathfrak{n}]$.

**Lemma 1.2.** If $\dim \mathfrak{n} < \infty$ and if $\mathfrak{n}$ is nilpotent then $H^j(\mathfrak{n}, N(W)) = (0)$ for all complex vector spaces $W$ and all $k > 0$.

**Proof.** We first assume that $\dim \mathfrak{n} = 1$. Let $X$ be a basis of $\mathfrak{n}$. We must show that $H^j(\mathfrak{n}, N(W)) = (0)$. From the definition of the cohomology, this means that we must show that $XN(W) = N(W)$. If $f \in N(W)$ then $f$ is determined by its values $f(X^s)$. Since $f \in N(W)$, there exists $r$ such that $f(X^j) = 0$ for $j \geq r$. Define $g(1) = 0$ and $g(X^{s+1}) = f(X^s)$. Then $Xg = f$. This proves the result in this case.

Suppose that the result has been proved if $\dim \mathfrak{n} = k$. Assume that $\dim \mathfrak{n} = k + 1$. Let $X \in \mathfrak{n}$ be such that $\mathfrak{n} = CX \oplus \mathfrak{n}_1$ with $[X, \mathfrak{n}] \subset \mathfrak{n}_1$ and $\mathfrak{n}_1$ is a Lie subalgebra. Then $U(\mathfrak{n}_1) = \oplus_{k \geq 3} X^k U(\mathfrak{n}_1)$. This means that as an $\mathfrak{n}_1$-module, $N(W)$ is a countable direct sum of modules of the form $\text{Hom}_C(U(\mathfrak{n}_1), W)[\mathfrak{n}_1]$. Thus the inductive hypothesis implies that $H^j(\mathfrak{n}_1, N(W)) = (0)$ for $j > 0$. We can now apply the Hochschild-Serre spectral sequence (cf. [BW, I, 6.5]) to find that $H^j(\mathfrak{n}, N(W)) =$
$H^j(\mathfrak{n}/\mathfrak{n}_1, N(W)^n)$. Clearly, $N(W)^n = \text{Hom}_C(U(\mathfrak{n}/\mathfrak{n}_1), W)$. Thus the one dimensional case now implies the result.

We assume that $\mathfrak{n}$ is as in the previous Lemma.

**Proposition 1.3.** Let $M \in N$. If $H^j(\mathfrak{n}, M) = (0)$ then $M$ is isomorphic with $N(M^n)$. In particular, $H^j(\mathfrak{n}, M) = (0)$ for $j > 0$.

§ 2. The vanishing theorems

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{R}$. Let $\theta$ be a Cartan involution for $\mathfrak{g}$. We will take this to mean that there exists a symmetric, $\mathfrak{g}$ and $\theta$-invariant, bilinear form $B$ on $\mathfrak{g}$ such that $B(X, \theta X) < 0$ for $X \in \mathfrak{g}$, $X \neq 0$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and set $\mathfrak{m} = \mathfrak{p} \cap \theta \mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$. Let $\mathfrak{g}(\mathfrak{m})$ denote the center of $\mathfrak{m}$ and put $\mathfrak{a} = \{X \in \mathfrak{g}(\mathfrak{m}) | \theta X = -X\}$. Then $\mathfrak{m} = \{X \in \mathfrak{g} | [X, a] = (0)\}$.

Let $\mathfrak{u} \subset \mathfrak{n}$ be an $\mathfrak{m}$-invariant subalgebra and let $\psi$ be a Lie algebra homomorphism of $\mathfrak{u}$ to $\mathfrak{C}$. Since $B$ induces a perfect pairing between $\mathfrak{u}_C$ and $\theta(\mathfrak{u}_C)$ there exists a unique $Y_\psi \in \theta \mathfrak{u}_C$ such that $B(X, Y_\psi) = \psi(X)$ for $X \in \mathfrak{u}$. We say that $\psi$ is non-degenerate if there exists $H \in \mathfrak{a}$ such that $\text{ad} H$ has integral eigenvalues and if we set $\mathfrak{g}_1 = \{X \in \mathfrak{g} | [H, X] = jX\}$ then the following two conditions are satisfied

1. $Y_\psi \in (\mathfrak{g}_C)_-$, $\mathfrak{g}_1 = \mathfrak{m}$, $\mathfrak{g}_2$ is contained in and generates $\mathfrak{u}$.
2. $\text{ad} Y_\psi : \mathfrak{u} \rightarrow \mathfrak{g}_C$ is injective.

If $\mathfrak{u} = \mathfrak{n}$ then this is the definition of $[L]$ of admissible. We will give some examples in Section 4 (see also [L]). The easiest way to guarantee the crucial condition (2) is to find $X \in (\mathfrak{u}_C)_2$ such that $[X, Y_\psi] = H$.

Fix such a non-degenerate $\psi$. Set $Y = Y_\psi$. If $X \in \mathfrak{g}_C$ then set $\bar{X}$ equal to the complex conjugate of $X$ relative to $\mathfrak{g}$. If $u, v \in \mathfrak{u}_C$ then set $(u, v) = B([\theta \bar{Y}, \theta \bar{v}], u, v)$. If $u \neq 0$ then $(u, u) = -B([\bar{Y}, u], \theta [Y, u]) > 0$ since $[Y, u]$ $\neq 0$.

Let $j > 0$ be fixed for the moment. Let $X_1, \ldots, X_\mathfrak{g}$ be a basis of $(\mathfrak{g}_C)_j \cap \mathfrak{u}_C$ such that $(X_1, X_\mathfrak{g}) = \delta_{1j}$. Set $Z_k = [\theta \bar{Y}, \theta X_k] \in (\mathfrak{g}_C)_{-j+2}$. Then $B([Z_k, X_k], Y) = (X_k, X_k) = \delta_{k, j}$. Fix $X_0 \in (\mathfrak{g}_C)_j (\supset \mathfrak{u}_C)$ such that $\psi(X_0) = 1$. Then $[Z_k, X_k] = \delta_{1j}X_0 + X_{ik}$ with $X_{ik} \in (\mathfrak{g}_C)_2$ and $\psi(X_{ik}) = 0$. Set $Q_j = \sum_i Z_i (X_i - \psi(X_i))$. Notice that $Q_j$ is independent of the choice of the $X_i$.

If $V$ is a $\mathfrak{u}$-module then we can define a new $\mathfrak{u}$-module structure on $V$ by tensoring with the one dimensional $\mathfrak{u}$-module $C_{-j}$ ($\mathfrak{u}$ acts by $\psi$ on $C$). We write $V_{p} = (V \otimes C_{-p})^p$ for $p \geq 0$ (see § 1 for notation). The following
Lemma is the key to all of the results in the paper. Since its proof is extremely complicated, we defer the proof to the appendix to this paper.

**Lemma 2.1.** Suppose that $V$ is a $g$-module such that $V = \bigcup_{n \geq 0} V_n$ (i.e. $(V \otimes C_{-q}) = (V \otimes C_{-q})[m]$). Fix $j > 0$. If $v \in V_k$ then there exist (not necessarily distinct) non-negative integers $n_1, \ldots, n_q$ depending only on $k$ such that

$$\prod_{i=1}^q (Q_i + n_i I)v = 0.$$ 

The next result implies a generalization of a Theorem of Kostant, Lynch. We maintain the above notation.

**Theorem 2.2.** Let $V$ be a $g$-module such that $V = \bigcup_{n \geq 0} V_n$ as a $u$-module. Then $H^i(u, V \otimes C_{-q}) = (0)$ for $j > 0$.

**Proof.** Proposition 1.4 implies that it is enough to prove that $H^i(u, V \otimes C_{-q}) = (0)$. Set $u_{i,j} = u \cap (\mathfrak{u}_c)_{i,j}$. Then $u_c = \bigoplus_{i,j} u_{i,j}$ and $[u_{i,j}, u_{k,l}] \subset u_{i,j+k,l}$. Thus if $\omega \in Z^i(u_c, V_p \otimes C_{-q})$ then $\omega(u_{i,j}) \subset V_{p-i+1}$ for $i \geq 1$.

**Scholium.** Let $\omega \in Z^i(u_c, V_p \otimes C_{-q})$. Suppose that $j \geq 1$ and that $\omega(u_{i,j}) \subset V_{p-i}$ for $i > j$. Then there exists $v \in V_{p+1}$ such that $(\omega - dv)(u_{1,i}) \subset V_{p-i}$ for $i \geq j$. Here, as usual, $dv(X) = Xv$.

We first show how to prove the theorem using the Scholium and then we will prove the Scholium. Since $u_{2r+2} = (0)$ for $r$ sufficiently large, $\omega(u_{2r+2}) \subset V_{p-r-1}$. Hence there exists $v_1 \in V_{p+1}$ such that $(\omega - dv_1)(u_{1,i}) \subset V_{p-i}$ for $i \geq r$. Now, $\omega - dv_1 \in Z^i(u_c, V_p \otimes C_{-q})$. Hence there exists $v_2 \in V_{p+1}$ such that

$$(\omega - dv_1 - dv_2)(u_{i,j}) \subset V_{p-i}, \quad i \geq r - 1.$$ 

Continuing in this manner we can find $v \in V_{p+1}$ such that

$$(\omega - dv)(u) \subset V_{p-1}. $$

Now use the same argument with $p$ replaced by $p-1$ and $\omega$ replaced by $\omega - dv$. After $p$ steps we find that $\omega \in B^i(u_c, V_0 \otimes C_{-q}) (V_0 = (0))$. Thus the Scholium implies the theorem.

The proof of the Scholium will take some preparation.

(i) If $Z \in \mathfrak{u}_{2j}, j \geq 1$ then $(Z - \psi(Z))V_p \subset V_{p-j}$.

Indeed, $Z - \psi(Z)$ is a linear combination of commutators of elements of the form $Y_1 - \psi(Y_1), \ldots, Y_j - \psi(Y_j)$ with $Y_i \in \mathfrak{u}_2$. Since $(X - \varphi(X))V_p \subset V_{p-1}$ for $X \in \mathfrak{u}_2$, (i) follows.

(ii) If $Z \in \mathfrak{g}_{-2j}, j \geq 0$ then $ZV_p \subset V_{p+j+1}$.

We prove this by induction on $j$ and for each $j$ by induction on $p$. If
Assume that \( m \subset V_{p+1} \) (don't forget that \( m = Z \)). If \( u \in V_{p+1}, X \in \mathfrak{z}, Z \in m \) then \((X - \psi(X))Zu = [X, Z]u + Z(X - \psi(X))u\). Now \([X, Z] \in \mathfrak{z}\) so \([X, Z]V_{p+1} \subset V_{p+1}\). Also, \((X - \psi(X))u \in V_{p}\). So, \(Z(X - \psi(X))u \in V_{p+1}\) by hypothesis. This implies that \((X - \psi(X))ZV_{p+1} \subset V_{p+1}\). Hence, \(ZV_{p+1} \subset V_{p+2}\). This proves (ii) for \( j = 0 \) and all \( p \).

Assume (ii) for \( j \) (fixed) and all \( p \). We prove it for \( j + 1 \) and all \( p \) by induction on \( p \). Let \( Z \in (\mathfrak{g}_C)_{-2j-2} \). Clearly, \( ZV_0 = 0 \). Assume that \( ZV_p \subset V_{p+j+2} \). If \( u \in V_{p+1}, X \in \mathfrak{z} \) then \((X - \psi(X))Zu = [X, Z]u + Z(X - \psi(X))u\). Now \([X, Z] \subset (\mathfrak{g}_C)_{-2} \). Hence, \([X, Z]V_{p+1} \subset V_{p+j+2}\). \((X - \psi(X))u \in V_p\) so \(Z(X - \psi(X))u \in V_{p+j+2}\). We therefore see that \((X - \psi(X))Zu \in V_{p+j+2}\). Hence \(Zu \in V_{p+j+2}\). This completes the proof of (ii).

We are (finally) ready to prove the Scholium. Let \( \omega \) satisfy the hypothesis of the Scholium. Set \( u_i = -\Sigma Z_i \omega(X_i) \) (as above for \( j \)). Note that \( Z_i \in (\mathfrak{g}_C)_{-2j-2} \). We calculate

\[
dv_i(X_i) = -\Sigma Z_i \omega(X_i)Z_i \omega(X_i)
\]

\[
= -\Sigma [X_i, Z_i] \omega(X_i) - Z_i \omega(X_i) - Z_i \omega(X_i) - Z_i \omega(X_i)
\]

\[
= X_i \omega(X_i) - Z_i \omega(X_i) - Z_i \omega(X_i) - Z_i \omega(X_i)
\]

\[
- Z_i \omega([X_i, X_i]).
\]

Here we have used the relation \([Z_i, X_i] = \delta_{ik} X_i + X_{ik} \) (don't forget \( \psi(X_{ik}) = 0 \)) and the cocycle condition on \( \omega \). If \( X \in (\mathfrak{z}_C) \) then \( \omega(X) \subset V_{p-j} \). Hence \( (X_0 - 1) \omega(X_i) \omega(X_i) \subset V_{p-j} \). We also note that \( \omega([X_i, X_i]) \subset V_{p-j} \) by our hypothesis. Thus \( Z_i \omega([X_i, X_i]) \subset V_{p-j} \). We have therefore shown that

(iii) \((\omega - dv_i)(X_i) = u_k + Q_j \omega(X_k) \) with \( u_k \in V_{p-j} \).

We observe that

(iv) \( Q_j V_p \subset V_p \) all \( p \).

Indeed, \((X_i - \psi(X_i))V_p \subset V_p \) by (i) and \( Z_i V_{p-j} \subset V_p \) by (ii).

(v) If \( X \in (\mathfrak{z}_C) \), \( i \geq j \), then \( dv_i(X) \in V_{p-j} \).

Indeed, \( dv_i(X) = -\Sigma_i [X_i, Z_i] \omega(X_i) - \Sigma_i Z_i \omega(X_i) - Z_i X_i \omega(X_i) = -\Sigma_i [X_i, Z_i] \omega(X_i) - Z_i Z_i \omega(X_i) - \Sigma_i Z_i \omega([X_i, X_i]). \)

\([X_i, Z_i] \in (\mathfrak{z}_C)^{2j+2}, \omega(X_i) \in V_{p-j} \) so \( [X_i, Z_i] \omega(X_i) \in V_{p-j} \). \( \omega(X) \in V_{p-j} \) and \( Q_j V_{p-j} \subset V_{p-j} \). Finally, \( [X_i, X_i] \in (\mathfrak{z}_C)^{2j+2} \) so \( \omega([X_i, X_i]) \in V_{p-j} \). Hence \( Z_i \omega([X_i, X_i]) \in V_{p-j} \). (v) now follows.

(vi) implies that \( \omega - dv_i \) satisfies the same hypothesis as \( \omega \). We can thus iterate on (iii) to find

(vii) There exists for each \( q = 1, 2, \cdots, v_q \in V_{p+1} \) such that \( \omega - dv_q \) satisfies the hypothesis of the Scholium and

\[
(\omega - dv_q)(X_k) = u_{k,q} + Q_{j} \omega(X_k), \quad \text{with} \ u_{k,q} \in V_{p-j}.
\]
Lemma 2.1 implies that there exists a polynomial $f(X) = X^{k} Y_{i}^{r} (X + n_{i})$ with $n_{i} \geq 0$ such that $f(Q_{k}) \omega(X_{k}) = (0)$ for all $k$. Thus if we write $f(X)/f(1) = \Sigma_{j \geq 1} b_{j} X^{j}$ ($f(1) > 0$) then $\Sigma b_{j} = 1$. Hence

$$(\omega - d(\Sigma b_{j} X_{j}))(X_{k}) = \Sigma b_{i} u_{k} \in V_{p - 1}.$$ 

This completes the proof of the Scholium hence of the Theorem.

We now show how the above theory implies the theorem in the introduction. Let $Y \in \mathfrak{g}$ be a nilpotent element (i.e. $\text{ad} \ Y^{k} = 0$ for some $k > 0$). Then the Jacobson-Morosov theorem (cf. [W, 8, A. 4. 1]) implies that there exist $X, H \in \mathfrak{g}$ such that $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. In particular, $\text{ad} \ H$ is semi-simple with integral eigenvalues. Thus there exists an inner automorphism $g$ of $\mathfrak{g}$ such that $\theta g H = -H$. So replace $\theta$ by $g^{-1} \theta g$. Let $q_{j}$ be (as usual) the eigenspace for $\text{ad} \ H$ with eigenvalue $j$. Put $p = \bigoplus_{j > 0} q_{j}$, $m = \mathfrak{g}_{0}$, $n = \bigoplus_{j > 0} q_{j}$. Then $p$ is a parabolic subalgebra of $\mathfrak{g}$ with standard Levi decomposition $\mathfrak{p} = m \oplus n$ (i.e. $m = p \cap \mathfrak{g}$). Define $u$ to be the subalgebra of $\mathfrak{u}$ generated by $q_{0}$. If $z \in \mathbb{C} \setminus \{-1\}$ then set $\psi(x) = zB(Y, x)$ for $x \in \mathfrak{u}$. Then $\psi$ is non-degenerate. Now Theorem 2.2 implies the theorem of the introduction.

§ 3. The category $W_{\psi}$

Let $\mathfrak{g}$, $p = m \oplus n$, $u \subset n$, $\psi$, $Y \in \mathfrak{g}$, $H \in \mathfrak{a}$ be as in the definition of non-degenerate in the previous section. Let $W_{\psi}$ be the category of all $\mathfrak{g}$-modules, $V$, such that $V = \bigcup_{p > 0} V_{p}$ (see § 2 for notation). Note that $V_{1} = \{u \in V_{1} | Xu = \psi(X)u \}$ for $X \in u_{1}$. As in Section 2 we set $u_{2j} = \{X \in u \}$, $[H, x] = 2jX$. We assume that $u_{2d} = (0)$ but $u_{2d+1} = (0)$. Let $Q_{j}$ be defined as in Section 2.

Lemma 3.1. Let $V \in W_{\psi}$. If $j \geq 1$ then there exist integers $n_{i, k, j} > 0$, $k = 1, \ldots, p_{i}$, $i = 2, \ldots, d$ depending only on $j$ such that if

$$T_{j} = (Q_{1} + jI) \Pi_{k} (Q_{2} + n_{2, k, j} I) \Pi_{k} (Q_{3} + n_{3, k, j}) \cdots \Pi_{k} (Q_{d} + n_{d, k, j}).$$

Then $T_{j} V_{j+1} \subset V_{j}$.

Proof. Let for $r \geq 1$, $V_{j} = \{u \in V_{j} \mid (X - \psi(X))u \in V_{j-s-1} \}$ for $X \in u_{2s}$, $s \geq r$. We note that $V_{j} = V_{j-1}$.

1. $Q_{r} V_{j+r+1} \subset V_{j+r+1}$, $r \geq 1$.

Indeed, set $Q_{r} \Sigma_{j} Z_{i}(X_{i} - \psi(X_{i}))$ as in Section 2. If $X \in u_{2s}$, $s \geq r+1$ and if $v \in V_{j+r+1}$ then

$$XQ_{r} v = \Sigma_{j} [X, Z_{i}] (X_{i} - \psi(X_{i})) v + \Sigma_{j} Z_{i} [X, X_{i}] v + Q_{r} X v.$$ 

$[X, Z_{i}] \in u_{2(s-r+1)}$. Hence $[X, Z_{i}] (X_{i} - \psi(X_{i})) v \in V_{j-s-1}$. $[X, X_{i}] \in u_{2(s-r)}$. 

[END OF PAGE]
So \([X, X_j]u \in V_{j-s-r-1} \) \((s+r > s)\). Hence \(Z_k[X, X_j]u \in V_{j-s-1}\) by (ii) in the proof of Theorem 2.2. Finally, \(Xu \in V_{j-s-1}\) so \(Q_r Xu \in V_{j-s-1}\) by (iv) in the proof of Theorem 2.2.

(2) If \(r \geq 2\) then there exist integers \(m_i > 0\) depending only on \(j, r\) such that

\[ \Pi_i (Q_r + m_i I) V_{j+1}^r \subseteq V_j^r. \]

Let \(X \in u_{1r}\) and \(u \in V_j^r\). Then \(Xu = \sum_k XZ_k X_k \in Q_r Xu + \sum_k [X, Z_k] X_k + \sum_k Z_k [X, X_k]u\). Now, \([X, X_k] \in u_{1r}\) and since \(4r \geq r + 1\) this implies that \([X, X_i]v \in V_{j-s-1}\). Thus \(Z_k [X, X_i]v \in V_{j-s-1}\). Also, the definition of the \(Z_k, X_k\) implies that \(\sum_k [X, Z_k] X_k = -X_0 X + \sum_k u_k X_k\) with \(u_k \in u_{1r}\) and \(\psi(u_k) = 0\). Thus \(\sum_k [X, Z_k] X_k u = -Xu\) mod \(V_{j-s-1}\). Hence \(X(Q_r + I)u \equiv Q_r u\) mod \(V_{j-s-1}\). Thus we see that

\[ X(Q_r + I)^p u \equiv Q_r^p Xu \text{ mod } V_{j-s-1}. \]

Let \(n_1, \ldots, n_q\) be such that \(n_i \geq 0\) and \(\Pi_i (Q_r + n_i) V_j = 0\) (Lemma 2.1). These integers depend only on \(j\). Then

\[ X \Pi_i (Q_r + n_i + 1) I u \in V_{j-s-1}. \]

This proves (ii).

(iii) If \(u \in V_j^r\) then \((Q_r + j I)u \in V_j^r\).

Indeed, let \(Y_1, \ldots, Y_j \in u_{1r}\). Set for \(x \in u_{1r}\), \(x' = x - \psi(x)\). Then

\[ Y_1' \cdots Y_j' Q_r u = \sum_{i=1}^j \sum_k Y_i' \cdots Y_i' - [Y_i, Z_k] X_k Y_i' \cdots Y_j' u \]

\[ + \sum_{i=1}^j \sum_k Y_i' \cdots Y_i' - Z_k Y_i, X_k Y_i' \cdots Y_j' u. \]

We assert that \([Y_i, Z_k] = -(Y_i, X_k) X_k + u_{i,k}\) with \(u_{i,k} \in u_{1r}\) and \(\psi(u_{i,k}) = 0\) thus the first term on the right hand side of (10) contributes

\[ -j Y_1' \cdots Y_j' u. \]

The second term is zero. To see this we observe as in the proof of (ii) that if \(x \in u_{1r}\) then \(x' V_j^r \subset V_{j-1}^r\). Thus \([Y_i, X_i] Y_i' \cdots Y_j' u \in V_{j+1 - (j-I) - 1} = V_{j-1}^r\). Since \(Z_k V_{j-2} \subset V_{j-1}\) the assertion follows. This proves (iii).

The Lemma is a direct consequence of (ii) and (iii).

**Corollary 3.2.** If \(u \in V_{j+1}\) with \(j \geq 0\) then \(T_1 \cdots T_j u \in V_1\).

Let \(F\) be a finite dimensional \(g\)-module. \(u\) acts nilpotently on \(F\). Thus if \(V \in W_q\) then \(V \otimes F \in W_q\). Let \(k\) be the largest eigenvalue of \(H\).
on $F$. We set $F^j = \{ f \in F \mid Hf = (k - j)f \}$. Let $V \in W_\sigma$. Then $V \otimes F^j \subseteq (V \otimes F)_{r+1}$ if $j = 2r$ or $j = 2r + 1$. Fix $F$ and $V$.

Lemma 2.3. Let $f \in F^j, v \in V_j$. If $r \geq 1$ and $1 \leq j_1, \ldots, j_r \leq d$ then

$$Q_{j_1} \cdots Q_{j_r}(v \otimes f) \in V \otimes \left( \sum_{i < j} F^i \right).$$

Proof. As usual, set $Q_r = \sum_i Z_{i, r} X'_{i, r}$ ($X' = X - \psi(X)$). Then $Q_{j_1} \cdots Q_{j_r}(v \otimes f)$ is a sum of terms of the form

$$(*) \quad Y_1 \cdots Y_r v \otimes U_1 \cdots U_r f$$

with each pair $(Y_i, U_i)$ in one of the following four forms:

1. $(Z_{k, j_1} X_{k, j_2})$
2. $(X'_{k, j_1}, Z_{k, j_1})$
3. $(Z_{k, j_1} X'_{k, j_1}, I)$
4. $(I, Z_{k, j_1} X_{k, j_1})$.

We show that each term as in $(*)$ above satisfies the conclusion of the Lemma. Fix such a term. Let for $i = 1, 2, 3, 4$, $S_i = \{ j \mid (Y_i, U_i) \text{ is as in (i)} \}$. Then $Y_1 \cdots Y_r v \in V_p$ with $p = 1 + \sum_{i \in S_1} j_i - \sum_{i \in S_2} j_i$. Thus if $Y_1 \cdots Y_r v \neq 0$ then $\sum_{i \in S_1} j_i \geq \sum_{i \in S_2} j_i$. Also, $U_1 \cdots U_r f \in F^{r+s}$ with $s = -2 \sum_{i \in S_1} j_i + 2 \sum_{i \in S_2} (j_i - 1) - 2 \sum_{i \in S_3} j_i$. Thus if $Y_1 \cdots Y_r v \otimes U_1 \cdots U_r f \neq 0$ then $s \leq -2 |S_1| - 2 |S_4|$. Thus if $s \geq 0$ then $S_1$ and $S_4$ are empty and $s \leq -2 |S_1|$. If $S_1$ is empty also then $Y_1 \cdots Y_r v = 0$ (indeed, $Y_1 v = 0$). Thus $s < 0$ if $Y_1 \cdots Y_r v \otimes U_1 \cdots U_r f \neq 0$. This implies the Lemma.

Let for $v \in V_i, f \in F^j, j = 2r$ or $2r + 1$, $\Gamma_i(v \otimes f) = v \otimes f$ for $r = 0$. Set $c_r = r! |\prod_{t \geq 2, i \leq r, t \neq i} n_{i, t, j} |^r \neq 0$ (see Lemma 3.1). Set $\Gamma_r(v \otimes f) = c_r^{-1} \Gamma_1 \cdots \Gamma_r(v \otimes f), r > 0$. We define a linear map, $\Gamma$, from $V_i \otimes F$ to $V \otimes F$ by $\Gamma(v \otimes \Sigma f_i) = \Sigma \Gamma_i(v \otimes f_i)$ ($f_i \in F^j$).

Theorem 3.4. (1) $\Gamma(V_i \otimes F) \subseteq (V \otimes F)_1$.
(2) $\Gamma$ defines a linear isomorphism of $V_i \otimes F$ onto $(V \otimes F)_1$.

Proof. (1) is just a restatement of Corollary 3.2.

We now prove that $\Gamma$ is injective. Let $f_1, \ldots, f_q$ be a basis of $F$ with $f_i \in F^{r_i}$ and $r_1 \leq \ldots \leq r_q$. Let $s_1 < \ldots < s_m$ be the distinct $r_i$ in order. Let $u \in V_i \otimes F$, $u = \Sigma v_i \otimes f_i$, $v_i \in V$. Assume that $u \neq 0$, we show that $\Gamma u \neq 0$. Let $i$ be the largest index such that $v_i \neq 0$. Let $r_i = s_k$. Then

$$u \equiv \sum_{r_j < r_i} v_j \otimes f_j \mod \sum_{r_j \leq r_i} v \otimes F^{r_j}.$$
Lemma 3.3 implies that
\[ \Gamma u \equiv \sum_{r, j \geq s_k} v_j \otimes f_j \mod \sum_{s_k < s_h} V \otimes F^{s_h}. \]

Hence \( \Gamma u \neq 0 \). To complete the proof we must show that \( \Gamma \) is surjective.

Let \( u \in (V \otimes F)_s \). Then \( u = \sum_i v_i \otimes f_i \) with \( v_i \in V \). If \( X \in \mathfrak{u} \) then
\[ 0 = (X - \psi(X))u = \sum_i [(X - \psi(X))v_i \otimes f_i + v_i \otimes Xf_i]. \]

Suppose that \( v_i = 0 \) for \( i > p + 1 \) and that \( v_p \neq 0 \). Set \( r_p = s_k \). Then
\[ 0 = (X - \psi(X))u \equiv \sum_{r, j = s_k} (X - \psi(X))v_r \otimes f_j \mod (V \otimes \sum_{s_k < s_h} F^{s_h}). \]

Thus \( (X - \psi(X))v_r = 0 \) if \( r = s_k \). As above,
\[ \Gamma \sum_{r, j = s_k} v_j \otimes f_j \equiv \sum_{r, j = s_k} v_j \otimes f_j \mod \sum_{s_k < s_h} V \otimes F^{s_h}. \]

So
\[ u - \Gamma \sum_{r, j = s_k} v_j \otimes f_j \in \sum_{s_k < s_h} V \otimes F^{s_h}. \]

If we iterate this argument we will find that after a finite number of stages that \( u = \Gamma w \) for some \( w \in V \otimes F \).

Note. The crucial point in this Theorem is that the formula for \( \Gamma \) depends on the structure of \( F \) and on \( \psi \) but not on \( V \). For the case \( n = \mathfrak{u} \) and without the explicit formula for \( \Gamma \) this result can be found in [L].

§ 4. Examples

In this section we give some examples of \( \mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{u} \) with \( \psi \) non-degenerate on \( \mathfrak{u} \). We retain the notation of Section 2.

Lemma 4.1. Let \( \mathfrak{p} \) be a minimal parabolic subalgebra of \( \mathfrak{g} \) and let \( \psi \) be a Lie algebra homomorphism of \( \mathfrak{n} \) to \( \mathfrak{c} \). Let \( \Delta \) be the set of simple roots of \( \mathfrak{a} \) on \( \mathfrak{n} \). Then \( \psi \) is non-degenerate if and only if \( \psi|_{d\mathfrak{a}} \neq 0 \) for \( \lambda \in \Delta \).

Note. This Lemma implies that in the case of minimal parabolic subalgebras the notion of non-degenerate coincides with that of "generic".

Proof. Assume that \( \psi \) is non-degenerate. \( Y = Y_\alpha = \sum_{i \leq 1} Y_i \) with \( Y_\chi \in \mathfrak{i}(\mathfrak{g}_\mathfrak{c})_\alpha \). Suppose that \( \mathfrak{h} \in \mathfrak{a} \) and that \( \mathfrak{m} = \{ X \in \mathfrak{g} | [H, X] = 0 \} \). We assume that \( \text{ad} H \) has integral eigenvalues and that \( \mathfrak{g}_\mathfrak{c} \supset \mathfrak{n} \) generates \( \mathfrak{n} \), \( [H, Y] = -2Y \) and \( \text{ad} Y \) is injective on \( \mathfrak{g}_\mathfrak{c} \). If \( Y_\alpha \neq 0 \) then \( \lambda(H) = 2 \). Let
\( \Phi = \Phi(p, a) \) be the root system of \( p \) with respect to \( a \). Then \( g_a = \bigoplus J_j \), the sum over \( \lambda \in \Phi \) such that \( \lambda(H) = 2 \). Let \( \Sigma = \{ \lambda \in A \mid Y_\lambda \neq 0 \} \) and let \( \Phi(\Sigma) \) denote the set of roots in the span of \( \Sigma \). If \( \lambda \in \Phi - \Phi(\Sigma) \) then \( [Y, g_\lambda] = 0 \). Thus if \( \Sigma \neq A \) then \( g_a \) cannot generate \( n \). Hence if \( Y \) is non-degenerate then \( Y_\lambda \neq 0 \) for all \( \lambda \in A \).

Assume now that \( Y_\lambda \neq 0 \) for all \( \lambda \in A \). Set \( X_\lambda = -\theta \bar{Y}_\lambda \). Let \( H = a \) be defined by \( B(h, H) = \lambda(h) \) for \( h \in a \). If \( a_i \in C \) for \( \lambda \in A \) then

\[
\left[ \sum_{i \in I} a_i X_i, Y \right] = -\sum a_i B(Y_i, \theta \bar{Y}_i) H_i.
\]

Now \( A \) is a basis for \( \text{span}_e \{ H_\lambda \mid \lambda \in A \} \). Thus we can choose the \( a_i \) such that

\[
\mu(-\sum a_i B(Y_i, \theta \bar{Y}_i) H_i) = 2, \quad \mu \in A.
\]

Set \( X = \Sigma a_i X_i, H = [X, Y] \). Then \( [H, X] = 2X, [H, Y] = -2Y \) and \( g_a = \bigoplus_{i,j} g_{i,j} \). The proposition now follows.

For the rest of this section we describe several techniques for finding examples of non-degenerate \( \psi \)'s. It is suggested that the reader consult the tables in [H, pp. 532–534 and p. 518].

The next class of examples were the original motivation for this paper. Let \( g \) be a simple Lie algebra over \( R \). Let \( p_0 \) be a minimal parabolic subalgebra of \( g \). Let the root system of \( g \) with respect to \( a_0 \) be of type \( C \), (resp. \( BC \)). Then there are linear functionals \( \varepsilon_i, \ldots, \varepsilon_r \) forming a basis of \( (a_0)^* \) such that the roots are \( \varepsilon_i - \varepsilon_j, i \neq j, \pm (\varepsilon_i + \varepsilon_j), i \leq j \) (resp. in addition \( \varepsilon_i \) for \( i = 1, \ldots, r \)). Let \( H_i \in a_0 \) be such that \( \varepsilon_i(H_i) = \delta_{ij} \). Let \( Y_i \in g_{-\varepsilon_i}, i = 1, \ldots, r \) and put \( Y = \Sigma Y_i \). We may choose \( X_i \in g_{\varepsilon_i} \) such that \( [X_i, Y_i] = 0 \). Since \( 2\varepsilon_i - 2\varepsilon_j \) is not a root for \( i \neq j \) we see that \( [X, Y] = H, [H, X] = 2X, [H, Y] = -2Y \). We choose \( p = m \oplus \eta \) with \( m = \{ x \in g \mid [H, X] = 0 \}, \eta = \bigoplus_{i \leq j} \sum_{\varepsilon_i \leq \varepsilon_j} \bigoplus_{i \leq j} \varepsilon_{i,j} \). Set \( u = \bigoplus_{i \leq j} \bigoplus_{i \leq j} \varepsilon_{i,j} \). Then \( u \) is a subalgebra of \( n \) and \( u = g_\varepsilon \). If \( \psi \) is defined by \( \psi(x) = B(x, Y) \) for \( x \in u \) then all the conditions for non-degeneracy are satisfied.

We list the examples that are given by this construction.

1. The subclass of examples where \( (g, f) \) is a Hermitian symmetric pair. Then \( p \) is the so called Shilov boundary parabolic. The examples are

   (1) \( \check{so}(p, q), p \geq q \geq 1 \) which is \( BC_q \) if \( p > q, C_q \) if \( p = q \).
   (2) \( \check{sp}(n, R), C_n \).
   (3) \( \check{so}(n, 2), C_2 \).
   (4) \( \check{so}^*(2n), C_n \) if \( n \) is even, \( BC_n \) if \( n \) is odd.
   (5) The Hermitian symmetric real form of \( E_6, BC_2 \).
   (6) The Hermitian symmetric real form of \( E_7, C_3 \).

The other examples from this construction are
2. \( \mathfrak{sp}(q, q), p \geq q \geq 1, BC_q \) unless \( p = q \) in this case \( C_q \).

3. The rank one real form of \( F_4, BC_4 \).

Suppose now that the root system of \( \mathfrak{g} \) with respect to \( \alpha_0 \) is type \( A_{2r+1} \). Then there exists an isometric imbedding of \( \alpha_0 \) into \( \mathbb{R}^{2r+2} \) as the hyperplane \( \{ x | \Sigma_{i} x_{i} = 0 \} \). Set \( \varepsilon_{i}(x) = x_{i} \). Then the roots are \( \varepsilon_{i} - \varepsilon_{j}, i \neq j \). We take to be the parabolic subalgebra defined as follows. Let \( H \in \alpha_0 \) be the element in \( \alpha_0 \) whose image in \( \mathbb{R}^{2r+2} \) is \( (1, \ldots, 1, 1, \ldots, 1, -1) \) where the last \( 1 \) is in the \( r+1 \)th position. Set \( \mathfrak{m} = \{ x \in \mathfrak{g} | [H, X] = 0 \} \), \( \mathfrak{n} \) is the eigenspace for \( \text{ad} H \) corresponding to eigenvalue \( 2 \). Then \( \beta_{i} = \varepsilon_{i} - \varepsilon_{i+1}, \beta_{i} \neq 0 \) if \( i \neq j \). Let \( Y_{i} \in \theta(\mathfrak{n}_{0}) \beta_{i} = 0 \) and set \( Y = \Sigma_{i} Y_{i} \). Put \( \psi(x) = B(Y, x) \) for \( x \in \mathfrak{n} \). Then we leave it to the reader to check that \( \psi \) is non-degenerate. Here are the examples that come from this method.

4. \( \mathfrak{sl}(2n, \mathbb{R}) \).

5. \( \mathfrak{sl}(2n, \mathbb{C}) \).

6. The real form of \( E_6 \) with maximal compactly imbedded subalgebra of type \( F_4 \).

There are also two examples corresponding to \( D_n \). Assume that the root system of \( (g, \alpha_0) \) is of type \( D_{2n} \). Then the roots are given by \( \pm \varepsilon_{i} \pm \varepsilon_{j} \) for \( 1 \leq i \neq j \leq 2n \). Choose the simple roots to be \( \varepsilon_{1} - \varepsilon_{2}, \cdots, \varepsilon_{n-1} - \varepsilon_{n}, \varepsilon_{2n-1} + \varepsilon_{2n} \). We take for \( \mathfrak{p} \) the parabolic subalgebra corresponding to \( \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{2n-1} - \varepsilon_{2n} \). Then the roots of \( \alpha_0 \) on \( \mathfrak{n} \) are \( \varepsilon_{i} + \varepsilon_{j} \) for \( i \neq j \). Choose \( Y_{i} \) to be a nonzero element of \( \mathfrak{g}_{-\varepsilon_{i}-\varepsilon_{i+1}} \) for \( i = 1, \cdots, n \) and \( Y = z(\Sigma_{i} Y_{i}) \) with \( z \neq 0 \). Then it is easily checked that there exist for each \( i = 1, \cdots, n \), \( X_{i} \in \mathfrak{g}_{\varepsilon_{i}+\varepsilon_{i+1}} \) such that if \( X = z^{-1}(\Sigma_{i} X_{i}) \) then \( X, Y, H \) is a standard basis of a three dimensional simple Lie algebra (TDS) and \( \text{ad} H|_{\mathfrak{n}} = 2I, \mathfrak{m} = \{ x \in \mathfrak{g} | [H, x] = 0 \} \). In this case we get the following example.

7. \( \mathfrak{so}(2n, 2n) \).

For \( \mathfrak{sl}(n, \mathbb{R}) \) (or \( \mathfrak{sl}(n, \mathbb{C}) \)) examples abound. We now give a method which probably gives all examples for \( \mathfrak{sl}(n, \mathbb{R}) \). Let \( H, X, Y \) be a standard basis of a TDS with \( H, X, Y \in M_{n}(\mathbb{R}) \) we assume, as we may, that \( H \) is symmetric. We assume that all of the eigenvalues of \( H \) are congruent mod 2. We take for \( \mathfrak{p} \) the direct sum of the eigenspaces for the non-negative eigenvalues of \( \text{ad} H \). Then \( \mathfrak{n} \) is generated by the 2-eigenspace for \( \text{ad} H \). We take \( \psi(x) = z \text{tr} Yx \) for \( x \in \mathfrak{n} \) and \( z \in \mathbb{C} - \{ 0 \} \).

The last class of examples that we will describe is of a somewhat more sophisticated nature. Let \( G \) be a connected semi-simple algebraic group defined over \( k \), a subfield of \( \mathbb{R} \). We look upon \( G \) as its complex points hence as a Lie group over \( \mathbb{C} \). Let \( P \) be a minimal parabolic subgroup of \( G \) defined over \( k \) and let \( S \) be a maximal \( k \)-split torus of \( P \). Let \( \phi \) be the root system of \( (G, S) \) and let \( \phi^{+} \) be the positive roots corresponding to \( P \). We assume that \( \phi \) is reduced (i.e. if \( \alpha \in \phi \) then \( 2\alpha \in \phi \)).
Let $\Delta \subset \Phi^+$ be the set of simple roots.

Let for $\alpha \in \Delta$, $E_\alpha$ be a one dimensional subgroup of the unipotent subgroup of $G$ corresponding to $\alpha$ that is invariant under conjugation by $S$ and defined over $k$. Let $V$ be the subgroup of $G$ generated by the $E_\alpha$ for $\alpha \in \Delta$. Then it is shown in [BT, Theorem 7.2] that there is a unique $k$-split reductive algebraic group $F \subset G$ containing $SV$ with $S$ as its maximal split torus.

Let $G = G_{\mathbb{R}}$, $P = P_{\mathbb{R}}$, $A = (S_{\mathbb{R}})^0$ (upper $o$ indicates identity component). Let $N$ be the unipotent radical of $P$ and $N = N_{\mathbb{R}}$. Let $\overline{P}$ be the opposite parabolic to $P$ and let $\overline{N}$ be its unipotent radical put $\overline{N} = N_{\mathbb{R}}$.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{g}_\mathbb{R} = \mathfrak{g}$. $\mathfrak{g}_\alpha$ is a $k$-form of $\mathfrak{g}$ and the Killing form of $\mathfrak{g}$ is defined over $k$. Let $\psi$ be a Lie algebra homomorphism of $\mathfrak{n}$ to $\mathfrak{c}$. Then there exists $Y = Y_\alpha \in \mathfrak{n}_\mathbb{R}$ such that $B(Y, x) = \psi(x)$. Write $Y_\alpha = \sum_{\alpha \in \Delta} Y_\alpha$ with $Y_\alpha \in (\mathfrak{n}_\alpha)_\mathbb{C}$. We assume that $\exp CY_\alpha$ is defined over $k$ and that $Y_\alpha \neq 0$ for all $\alpha \in \Delta$. Then (replacing $P$ by $\overline{P}$ in the above mentioned result of Borel and Tits) there exists a unique $k$-split (hence $\mathfrak{R}$-split) reductive subalgebra $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ containing and such that $\mathfrak{n}_\alpha \cap \mathfrak{g}_\alpha = CY_\alpha \cap \mathfrak{g}_\alpha$ is one dimensional over $\mathfrak{R}$. We can now argue as in the proof of Lemma 4.1 to find $X \in \oplus_{\alpha \in \Delta} (\mathfrak{n}_\alpha \cap \mathfrak{g}_\alpha)_\mathbb{C}$ such that $[X, Y] = H, H \in \alpha$ and $\text{ad} H|_{\mathfrak{n}_\alpha} = 2I, \alpha \in \Delta$.

§ 5. Some structural results

In this section we study the global structure of the non-degenerate functionals in the previous sections in preparation for the results on the holomorphic continuation of the corresponding generalized Jacquet integrals. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathfrak{R}$. Fix a Cartan involution $\theta$ of $\mathfrak{g}$. Let $G$ be a Lie group of inner type with finite center and Lie algebra $\mathfrak{g}$ (i.e. if $g \in G$ then $\text{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$). We assume that $\mathfrak{g}$ is one of the examples in Section 4 that corresponds to $C_n$, $A_{2r + 1}$ or $D_{2n}$. Let $\mathfrak{p}$ be the parabolic subalgebra described in those cases in Section 4 and let $\psi$ be a non-degenerate homomorphism of $\mathfrak{n}$ into $\mathfrak{R}$ as in that section. Let $P$ be the corresponding parabolic subgroup of $G$ (i.e. $P = \{ g \in G \mid \text{Ad}(g)p \subset \mathfrak{p} \}$). Let $H, X, Y$ be as in those cases. Put $M = \{ g \in G \mid \text{Ad}(g)H = H \}$. Set $M_\psi = \{ m \in M \mid \psi(\text{Ad}(m)x) = \psi(x) \text{ for } x \in \mathfrak{n} \}$. Set $N = \exp \mathfrak{n}$. Then $P = MN$ is a Levi decomposition of $P$.

**Proposition 5.1.** There exists $m \in M$ such that if we replace $\psi$ by $\psi \circ \text{Ad}(m)$ then there exists a finite subset $\Sigma \subset K$ such that

1. $G = \bigcup_{\alpha \in \Sigma} M_\alpha NwP$ disjoint union.

2. There is a unique element $w_\alpha \in \Sigma$ such that $M_\phi Nw_\phi P$ is open. Furthermore, $M_\phi Nw_\phi P = Nw_\phi P = Pw_\phi P$ is the unique open double coset of $P$. ```
If \( w \in \Sigma \) and \( w \neq w_o \) then \( \psi \) restricted to the Lie algebra of \( N \cap w^{-1}Pw \) is non-trivial.

Proof. In all of these cases \( n = g_2 = \{ x \in g_1 | [H, x] = 2x \} \). Thus \( g_{-2} = \theta n \). So \( g = g_{-2} \oplus m \oplus g_2 \). As a module (under \( ad \)) for the Lie algebra \( g \), with basis \( X, Y, H, g = g^0 \oplus g^2 \) with \( \theta \) acting trivially on \( g^0 \) and acting by a multiple of the irreducible three dimensional representation on \( g^2 \). We note that \( m_{\psi} = m \cap g^0 \). Set \( \sigma = \exp (\pi (X - Y)/2) \). Then a direct calculation in \( SL(2, R) \) implies that \( \text{Ad}(\sigma)x = x \) for \( x \in m_{\psi} \) and \( \text{Ad}(\sigma)x = -x \) for \( x \in m \cap g^2 \). There exists \( m \in M \) such that \( \text{Ad}(m\sigma^{-1}) \) commutes with \( \theta \). So replace \( \psi \) by \( \psi \circ \text{Ad}(m) \), \( Y \) by \( \text{Ad}(m)Y \), \( X \) by \( \text{Ad}(m)X \). \( M_{\psi} \) contains the identity component of the fixed point set of \( m \rightarrow \text{ad}m \sigma^{-1} \).

We now look at the \( C_n \) cases. Fix \( P_0 \) a minimal parabolic subgroup contained in \( P \). Let \( A_n \) be a standard split component of \( P_0 \) (\( \theta h = -h \) for \( h \in \alpha_0 \)). Let \( \epsilon_1, \ldots, \epsilon_n \) be as in Section 4. Then the Weyl group of \( (G, A_n) \) is the semi-direct product of the permutations of the \( \epsilon_i \), \( S_n \), and the sign changes on the \( \epsilon_i \), \( Z_n^* \). Let \( w_i \) be defined by \( w_i \epsilon_j = -\epsilon_j \) for \( j \leq i \) and \( w_i \epsilon_j = \epsilon_j \) for \( j > i \). The Bruhat lemma implies that

\[
G = P \cup \bigcup_{i=1}^n P w_i P.
\]

Set \( P_i^* = M \cap w_i P w_i^{-1} \). Then \( P_i^* \) is a minimal parabolic subgroup of \( M \). Thus Theorem 3 of the introduction of [M] implies that there exist a finite number of \( w_{ij} \in K \cap M \) such that \( M = \bigcup_{i,j} M_{w_{ij}} P_i^* \). Now

\[
P \cup \bigcup_{ij} N M_{w_{ij}} P w_{ij} P = P \cup \bigcup_{ij} N M_{w_{ij}} w_{ij} w_{ij}^{-1} P_i^* P w_i P = P \cup \bigcup_{i} N M w_i P = G
\]

by the above. Suppose that \( N M_{w_{ij}} P w_{ij} P \) is open then \( P w_i P \) is open so \( i = n \). We observe that \( N w_n P = P w_n P \). Thus in this case we may take all of the \( w_{in} = 1 \). This proves (1), (2). The explicit formulas for the \( w_{ij} \) in [M, Theorem 3] imply that \( \text{Ad}(w_{ij}) \subseteq \text{g}_{\epsilon_p + \epsilon_q + \epsilon_r} \) for all \( p, q \). Hence \( \psi \) is non-trivial on \( n \cap w_{ij} w_{ij}^{-1} w_{ij}^{-1} \) if \( j \neq n \). This completes the proof in the \( C_n \) case.

We now look at the \( A_{2r+1} \) case. Let \( P_0 \) be a minimal parabolic subgroup of \( G \). We use analogous notation to the previous case. However, this time the \( \epsilon_i, i = 1, \ldots, 2r+2 \) satisfy the relation \( \Sigma_i \epsilon_i = 0 \). The Weyl group is given by the permutations of the \( \epsilon_i \). Put \( w_j = \prod_{i=1}^{r+1} (i, r+1+i) \). The Bruhat lemma in this case implies that \( G = \bigcup_{j} P w_j P \). The rest of the argument is exactly the same as the case of \( C_n \).

The \( D_{2n} \) case is completely analogous and we leave it to the reader.

Let \( Y \in g \) be a nilpotent element and let \( X, Y, H \) be as in the end of Section 2. We assume that the eigenvalues of \( \text{ad} H \) are all even. Let \( G \) be a real reductive group of inner type with Lie algebra \( a g \) and let \( p \) be
the parabolic subalgebra of \( \mathfrak{g} \) given by the sum of the eigenspaces for the non-negative eigenvalues of \( \text{ad} \, H \). Let \( P \) be the corresponding parabolic subgroup of \( G \). We conjecture that a decomposition of the type of Proposition 5.1 is true for \( G \).

§ 6. Some Bruhat theory

Let \( G \) be a real reductive group of inner type. Let \( \theta \) be a Cartan involution of \( G \). Fix a minimal parabolic subgroup, \( P_\circ \), of \( G \). Let \( P \) be a parabolic subgroup of \( G \) with unipotent radical \( N_P \) and set \( M_P = \theta P \cap P \). Let \( A_P = \exp(\mathfrak{a}_P) \) with \( \mathfrak{a}_P = \{ X \in \mathfrak{g}(\mathfrak{m}_P) \mid \theta X = -X \} \). Set \( \alpha = \alpha_{P,\circ} \).

Let \( \eta \) be a homomorphism from \( N_P \) to \( S_i \) (the circle group). We set \( d_\eta = i\psi \). Put \( M_\circ = M_\circ = \{ m \in M_P \mid \eta(\text{mnm}^{-1}) = \eta(n), \, n \in N_P \} = \{ m \in M \mid \psi(\text{Ad}(m)X) = \psi(X), \, X \in \mathfrak{n}_P \} \). Throughout this section we will make the following assumptions about \( P \) and \( \eta \).

1. There exists a finite subset \( \Sigma \) of \( G \) such that \( G \) is the disjoint union \( \bigcup_{\omega \in \Sigma} P \omega M_\circ N_P \).
2. There is a unique element \( w_\circ \in \Sigma \) such that \( P \omega \circ M_\circ N \) is open in \( G \) and furthermore \( P w_\circ M_\circ N = P w_\circ N \) and \( w_\circ M_\circ w^{-1}_\circ = M_\circ \).
3. If \( w \in \Sigma \) and if \( w \neq w_\circ \) then \( \eta \) is not identically equal to 1 on \( N_P \cap w^{-1} P w \).

We note that if \( P = P_\circ \) and if \( \psi \) is non-degenerate (i.e. generic) then the above conditions are satisfied. Also, all of the \( C_n, A_{2r+1}, D_{2n} \) examples of Section 5 satisfy these conditions.

Put \( \circ M_P = \{ m \in M_P \mid \kappa(m)^2 = 1 \} \) for all continuous homomorphisms \( \kappa \) of \( M_P \) into \( \mathbb{R} - \{0\} \). Then \( M_P = A_P \circ M_P \) with unique expression. If \( \nu \in (\alpha(\nu))_C \) then we set \( \exp(h)^{\nu} = \exp(\nu(h)) \) for \( h \in \mathfrak{a}_P \). Set \( \rho_\circ = 1/2(\text{tr}(\text{ad} h|_{\mathfrak{a}_P})) \), \( h \in \mathfrak{a}_P \). Let \( (\sigma, H_\circ) \) be a finite dimensional representation of \( \circ M_P \). If \( \nu \in (\alpha(\nu))_C \) then we denote by \( \sigma_\circ \) the representation of \( P \) given by

\[
\sigma_\circ(man) = \sigma^{\nu_\circ}(m), \quad m \in \circ M_P, \quad a \in A_P, \quad n \in N_P.
\]

Let \( I_{\nu}^{\circ}(\sigma_\circ) \) denote the \( C^\infty \) induced representation of \( \sigma_\circ \) from \( P \) to \( G \). That is, \( I_{\nu}^{\circ}(\sigma_\circ) \) is the space of all smooth functions \( f \) from \( G \) to \( H_\circ \) such that \( f(pg) = \sigma_\circ(p)f(g) \) for \( p \in P, \, g \in G \). We endow \( I_{\nu}^{\circ}(\sigma_\circ) \) with the \( C^\infty \) topology. The action, \( \pi_{\nu}^{\circ}(\sigma_\circ) \), is given by \( (\pi_{\nu}^{\circ}(\sigma_\circ)(g)f)(x) = f(xg) \).

Let \( (\tau, V_\tau) \) be a finite dimensional representation of \( M_\circ N_P \) such that \( \tau(n) = \eta(n)I \) for \( n \in N_P \). We set \( \text{Wh}^\tau(I_{\nu}^{\circ}(\sigma_\circ)) \) equal to the space of all continuous linear maps, \( T \) from \( I_{\nu}^{\circ}(\sigma_\circ) \) to \( V_\tau \) such that \( T(\pi_{\nu}^{\circ}(\sigma_\circ)(m)f) = \tau(m)T(f) \) for \( f \in I_{\nu}^{\circ}(\sigma_\circ), \, m \in M_\circ N_P \). Let \( U_{\nu,\circ} \) be the space of all \( f \in I_{\nu}^{\circ}(\sigma_\circ) \) such that \( 
abla f \subseteq P \omega \circ N_P \circ M_\circ \).

**Theorem 6.1.** Let \( T \in \text{Wh}^\tau(I_{\nu}^{\circ}(\sigma_\circ)) \).
(1) If $T|_{V_{\sigma}, v} = 0$ then $T = 0$.
(2) There exists $S: H_{\sigma} \rightarrow V$, such that
   \begin{enumerate}
   \item $S_{\sigma}(m) = \tau(\varepsilon(m)) S$ for $m \in M_{\psi}$ and
   \item $T(f) = S \int_{H_{\sigma}} \sum_{p} \gamma(n)^{-1} f(w_{\phi} n) dn$ (here $dn$ is a fixed choice of invariant measure on $N_{p}$).
\end{enumerate}

**Proof.** The argument is standard Bruhat theory. If $f \in C_{\omega}^{\infty}(G)$ and if $u \in H_{\sigma}$ then set

\[ S_{\sigma}(f \otimes v)(g) = \left( \int_{P} \sigma(p)^{-1} f(pg) d_{c}(p) \right) v \]

where $d_{c}$ is a fixed choice of right invariant measure on $P$.

Then $S_{\sigma}$ defines a continuous linear map of $C_{\omega}^{\infty}(G) \otimes H_{\sigma}$ onto $I_{P, \sigma}$. If $T \in \text{Wh}(I_{P, \sigma})$ then set $\hat{T}(f)(v) = T(S_{\sigma}(f \otimes v))$. Then

\[ \hat{T}: C_{\omega}^{\infty}(G) \longrightarrow \text{Hom}_{C}(H_{\sigma}, V_{\tau}). \]

Put $L(g)f(x) = f(gx)$ (resp. $R(p)f(x) = f(xg)$). Then

\[ \hat{T}(L(p))f = \hat{T}(f) \cdot \sigma_{\varepsilon}(p), \]

\[ \hat{T}(R(m))f = \tau(m) \hat{T}(f) \]

for $p \in P$, $m \in M_{\psi} N$, $f \in C_{\omega}^{\infty}(G)$.

Set $\text{Wh}(I_{P, \sigma}) = \{ T \in \text{Wh}(I_{P, \sigma}) \mid \text{supp} \hat{T} \subset G - \text{Pw}_{\phi} N_{p} \}$. Then Bruhat theory (cf. [War, 5.3 p. 411]) implies that

\[ \text{dim } \text{Wh}(I_{P, \sigma}) \leq \sum_{w \in S_{\omega}} \text{dim Hom}_{\omega \otimes I_{P, \sigma} \otimes V_{\tau}}(H_{\omega}, S(g(\text{Ad}(w_{\phi}^{-1} \sigma + m_{\phi} + n_{p}) \otimes V_{\tau})). \]

Here, $w_{\phi} \in K$ is a representative for $w$, $S(\cdot)$ is the symmetric algebra. The proof uses (1) and (2). (3) implies that if $w \neq w_{\phi}$ then $\gamma|_{w^{-1} P \cap N_{P}} \neq 1$. Since $N_{p}$ acts trivially on $H_{\omega}$ and unipotently on $S(g/n_{P})$ we conclude that $\text{dim } \text{Wh}(I_{P, \sigma}) = (0)$.

The second part of the Theorem is also completely standard (cf. [War, Theorem 5.2.2.1]) so we leave it to the reader.

Set $(I_{P, \sigma})^{\gamma}$ equal to the space of all continuous linear functionals on $I_{P, \sigma}$, put $\text{Wh}(I_{P, \sigma})^{\gamma}$ equal the space of all $\lambda \in (I_{P, \sigma})^{\gamma}$ such that

\begin{enumerate}
   \item $\lambda(\pi_{P, \sigma}(n)f) = \gamma(n)\lambda(f)$, $n \in N_{p}$, $f \in I_{P, \sigma}$,
   \item Let $\pi'$ be the contragradient representation of $G$ on $(I_{P, \sigma})^{\gamma}$.
\end{enumerate}

Then $\pi'(M_{\psi})\lambda$ spans a finite dimensional space on which $M_{\psi}$ acts semi-simply.
Corollary 6.2. Assume that $\sigma$ is irreducible. Then $\dim \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi}) \leq \dim H_\sigma$.

Proof. Let $\tilde{M}_\psi$ denote the set of all equivalence classes of irreducible finite dimensional representations of $M_\psi$. Fix for $\gamma \in \tilde{M}_\psi$, a representative $V_\gamma \in M_\psi$. We decompose $\text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})$ into $M_\psi$-isotypic components as

$$\text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi}) = \sum_{\gamma \in \tilde{M}_\psi} \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})(\gamma).$$

Fix $\gamma \in \tilde{M}_\psi$. Let $V \subset \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})(\gamma)$ be a finite dimensional $M_\psi$-invariant subspace. We define $T: I_{\overset{\circ}{P}_\sigma,\psi} \to V^*$ by $T(f)(\lambda) = \lambda(f)$. Let $\mu \in \text{Hom}_{\tilde{M}_\psi}(V^*, V^*)$. We look upon $V^*$ as a $M_\psi N$-module with $N$ acting by $\gamma^{-1}I$. Then $\mu \circ T \in \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})$. Thus Theorem 6.1 implies that there exists $S(\mu) \in \text{Hom}_{\tilde{M}_\psi}(H_\sigma, V^*_\gamma)$ such that

$$\mu(T(f)) = S(\mu)\left(\int N \gamma^{-1}(n)f(w,n)dn\right), \quad f \in U_{\overset{\circ}{P}_\sigma,\psi}.$$ 

This implies that we have an injective linear mapping

$$S: \text{Hom}_{\tilde{M}_\psi}(V^*, V^*) \to \text{Hom}_{\tilde{M}_\psi}(H_\sigma, V^*).$$

Hence $\dim V^* \leq \dim \text{Hom}_{\tilde{M}_\psi}(H_\sigma, V^*) \dim V^*_\gamma$. This implies that

$$\dim \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})(\gamma) \leq \dim \text{Hom}_{\tilde{M}_\psi}(H_\sigma, V^*) \dim V^*_\gamma.$$

Hence

$$\dim \text{Wh}_\gamma(I_{\overset{\circ}{P}_\sigma,\psi})(\gamma) \leq \sum_{\gamma \in \tilde{M}_\psi} \dim \text{Hom}_{\tilde{M}_\psi}(H_\sigma, V^*_\gamma) \dim V^*_\gamma = \dim H_\sigma,$$

since $M_\psi$ is reductive in $M$.

§7. The holomorphic continuation of certain integrals

We retain the notation and assumptions of the previous section. We assume in addition that $\psi$ is non-degenerate. We write $M, N, A$ for $M_{\overset{\circ}{P}}$, $N_{\overset{\circ}{P}}, A_{\overset{\circ}{P}}$.

Let $I_{\overset{\circ}{P}_\sigma}$ denote the space of all $f \in C^\omega(K; H_\sigma)$ such that $f(pk) = \sigma(p)f(k)$ for $p \in P \cap K$ and $k \in K$. If $f \in K$ then set $f_{\sigma,p}(pk) = \sigma(p)f(k)$ for $p \in P$. Then the correspondence $f \to f_{\sigma,p}$ of $I_{\overset{\circ}{P}_\sigma}$ to $I_{\overset{\circ}{P}_\sigma,\psi}$ is a continuous isomorphism. We will thus consider $I_{\overset{\circ}{P}_\sigma,\psi}$ as $I_{\overset{\circ}{P}_\sigma}$ and under this identification $\pi_{\overset{\circ}{P}_\sigma,\psi}(g)f(k) = f_{\sigma,p}(kg)$.

Proposition 7.1. There exists a constant $q_\sigma \geq 0$ such that
\[
\int_N \eta(n)^{-1}f_{\nu,\sigma}(w, n)dn = J_{\nu,\sigma}(f) = J_\nu(f)
\]
converges absolutely and uniformly in compacta of \( \{ \nu \in a^*_c | \text{Re} (\nu, \alpha) > q_\sigma \} \) for all \( f \in I_{P, \sigma}^\omega \). If \( \nu \in a^*_c, \text{Re} (\nu, \alpha) > q_\sigma \) and if \( \lambda \in H_c^* \) then \( \lambda \circ J_\nu \neq 0 \). Hence in particular

\[
\dim \text{Wh}_\eta(I_{P, \sigma, \nu}^\omega) = \dim H_\nu.
\]

**Proof.** The first assertion follows from (by now) standard inequalities (due to Harish-Chandra) which we now recall. If \( g \in G \) then we write \( g = p(g)k(g) \). Neither \( p(g) \) nor \( k(g) \) are well defined. However, \( p(g)P \cap K \text{ and } P \cap Kk(g) \) are well defined. Fix a \( K \cap P \)-invariant inner product on \( H_\nu \). If \( f \in I_{P, \sigma}^\omega \), then

\[
\| \eta(n)^{-1}f_{\nu,\sigma}(w, n) \| = \| \sigma(p(w, n))f(k(w, n)) \| \leq \| f \|_\infty \| \sigma(p(w, n)) \|.
\]

We write \( p(g) = n(g)m(g)a(g) \) with \( n(g) \in N, m(g) \in O(M) \) and \( a(g) \in A \). Now the ambiguity is in \( m(g) \). We have

\[
\| \eta(n)^{-1}f_{\nu,\sigma}(w, n) \| \leq \| f \|_\infty \| \sigma(m(w, n)) \| a(w, n)^{\text{Re} \nu + \phi}.
\]

Let \( \log \) be the inverse map to \( \exp \) on \( n \). Fix on \( g \) the inner product \( \langle x, y \rangle = -B(x, \theta y) \). It is standard that (cf. [W, Proof of 4.5.6])

\[
\| \sigma(m(w, n)) \| \leq C(1 + \| \log (n) \|)^d
\]

for some positive constants \( C \) and \( d \).

It is thus enough to show that there exists \( q_\sigma \geq 0 \) such that if \( \nu \in a^* \) and \( (\nu, \alpha) > q_\sigma \) then

\[
\int_N a(w, n)^{\nu + \phi}(1 + \| \log (n) \|)^d dn < \infty.
\]

But this follows directly from (for example) [W, 4.5.4].

We now prove the second assertion. Let \( \varphi \in C_c^\infty(N) \) be such that

\[
\int_N \eta(n)^{-1}\varphi(n)dn = 1.
\]

If \( \nu \in H_\nu \) then set \( \beta_\nu(pw, n) = \varphi(n)v \) if \( g = pw, n, p \in P, n \in N \) and zero otherwise. Then \( \beta_\nu(p) \in I_{P, \sigma}^\omega \) and \( J_\nu(\beta_\nu(v)) = v \). This completes the proof.

Let \( (I_{P, \sigma})' \) denote the space of all continuous functionals on \( I_{P, \sigma}^\omega \) (which is endowed with the \( C^\infty \) topology). We will use the weak topology on \( (I_{P, \sigma})' \). We look upon \( \text{Wh}_\eta(I_{P, \sigma, \nu}^\omega) \) as a subspace of \( (I_{P, \sigma})' \).
Theorem 7.2. We keep the hypotheses of this section and the previous one. Let \( \mu \in H^s \) then \( \nu \to \mu \circ J_\nu \) extends to a weakly holomorphic map of \( a^s \) into \( (I^s)^\prime \). Furthermore, if \( \nu \in a^s \) then \( \dim \{ \mu \circ J_\nu \mid \mu \in H^s \} = \dim H_s \). In particular, \( \dim \text{Wh}_s ((I^s)^\prime) = \dim H_s \) for all \( \nu \in a^s \).

Proof. If \((\pi, H)\) is a smooth Frechet representation of \( G \) then we denote by \( H' \) the continuous dual of \( H \) and we write for \( g \in G \) (resp. \( X \in \mathfrak{g} \)) \( \pi'(g) = \lambda \circ \pi(g)^{-1} \) (resp. \( -\lambda \circ \pi(X) \)) for \( \lambda \in H' \). We set for \( X \in \mathfrak{u} \), \( \pi_\nu(X) = \pi(X) + \psi(X) \) and \( H'[\psi] = \{ \lambda \in H' \mid \pi_\nu(\mu)^k \lambda = 0 \} \) for some \( k \) and \( \pi_\nu(M_\psi) \lambda \) spans a finite dimensional space such that \( M_\psi \) acts semi-simply.

Notice that \( H'[\psi] \in W_{-\psi} \) in the notation of Section 3. We set \( \text{Wh}_s (H') = H'(\mathfrak{u}) \otimes \overline{C_\psi} = (H'[\psi])_s \). If \( F \) is a finite dimensional representation of \( G \) then (since \( \mathfrak{u} \) acts nilpotently on \( F \))

\[
(H \otimes F)[\psi] = H'[\psi] \otimes F'.
\]

Let \((\mu, F)\) be a finite dimensional irreducible representation of \( G \) such that

(i) \( M_\psi \) acts trivially on \( F^\psi = \{ \nu \in F' \mid (\theta_1 \psi) \nu = (0) \} \).

(ii) \( -\lambda \) is the action of \( \alpha \) on \( F^\psi \) (which is one dimensional) then

\[
(\lambda, \alpha) > 0, \quad \alpha \in \mathcal{F}(P, A).
\]

Such an \( F \) always exists (cf. [W, 4.A.2.3]). We note that

\[
I^\alpha_{\rho, \sigma} \otimes F \cong \text{Ind}_{\rho}^\alpha (\sigma \otimes \mu | _\rho)
\]

where \( \text{Ind}^\alpha \) indicates \( C^\alpha \) induction. Let \( H_0 \otimes F = X_1 \supset X_2 \supset \cdots \supset X_{d+1} = (0) \) be a \( P \)-invariant filtration with \( X_i / X_{i+1} = (\sigma_i)_{|_{\mathfrak{u}}} \) for appropriate \( \sigma_i, \nu_i \).

Here the action of \( P \) on \( H_0 \) is \( \sigma \). We assume as we may (see below) that \( \sigma = \sigma \) and \( \nu = -\lambda \). Writing \( I^\alpha = I^\alpha_{\rho, \sigma, \nu} \) then

\[
I^\alpha \otimes F = I^\alpha_1 \supset I^\alpha_2 \supset \cdots \supset I^\alpha_{d+1} = (0)
\]

with each of the spaces closed and invariant and \( I^\alpha_j / I^\alpha_{j+1} \cong I^\alpha_{\sigma, \nu, \psi} \).

Assume that \( \dim \text{Wh}_\nu ((I^\alpha_{\rho, \sigma, \nu})') = \dim H_\sigma \). Then

\[
\dim ((I^\alpha_{\rho, \sigma, \nu})'[\psi] \otimes F)_1 = \dim F \dim H_\nu
\]

by Theorem 3.4. Now \( ((I^\alpha_{\rho, \sigma, \nu})'[\psi] \otimes F)_1 |_{\sigma, \nu} \subset \text{Wh}_\nu ((I^\alpha_{\rho, \sigma, \nu})') \) so

\[
\dim ((I^\alpha_{\rho, \sigma, \nu})'[\psi] \otimes F)_1 |_{\sigma, \nu} \leq \dim H_\sigma
\]

by Corollary 6.2. Set \( W' = \{ \lambda \in ((I^\alpha_{\rho, \sigma, \nu})'[\psi] \otimes F)_1 \mid \lambda |_{\sigma, \nu} = 0 \} \). Then \( \dim W' \geq \dim F \dim H_\nu - \dim H_\sigma \). \( W' |_{\sigma, \nu} \subset \text{Wh}_\nu ((I^\alpha_{\rho, \sigma, \nu})') \) so \( \dim W' \geq \dim F \dim H_\nu - \dim H_\sigma \). If we continue in this way we find
that dim $W^\alpha \geq \dim H_*$. But $W_2 \subset \text{Wh}_\gamma ((I_{P,\alpha ,\nu ,-1})'').$ We have thus proved

(iii) If $\dim \text{Wh}_\gamma ((I_{P,\alpha ,\nu ,-1})') = \dim H_* \text{ then dim Wh}_\gamma ((I_{P,\alpha ,\nu ,-1})') = \dim H_*$. 

If $\nu \in \alpha_\nu^*$ then there exists $k > 0$ such that $\text{Re} (\nu + k \lambda , \alpha) > q_\alpha$ for all $\alpha \in \Phi (P, A)$. We have thus proved

(iv) If $\nu \in \alpha_\nu^*$ then $\dim \text{Wh}_\gamma ((I_{P,\alpha ,\nu ,-1})') = \dim H_*$. 

We now convert this into an analytic statement. Let

$$\delta : I_{P,\nu}^* \otimes F \longrightarrow I_{P,\nu}^* \otimes \kappa |_{P \cap K}$$

be defined by

$$\delta (f \otimes \nu) (k) = f(k) \otimes \mu (k) \nu.$$

Let $q$ be the natural projection of $F$ onto $F / n F$. We define

$$T_j : I_{P,\nu}^* \otimes F \longrightarrow I_{P,\nu}^* \otimes \kappa |_{P \cap K}$$

by $T_j (f \otimes \nu) = (I \otimes \zeta) (\delta (f \otimes \nu))$. We note that $T_j$ depends only on $F$ and $\sigma$ and

(iv) $T_j \circ (\pi_{P,\sigma} (g) \otimes \mu (g)) = \pi_{P,\sigma} (g) T_j$

(v) $T_j$ is surjective and continuous.

Set $\pi_\nu = \pi_{P,\nu}$. Suppose that we have proved the theorem for $\text{Re} (\nu, \alpha) > q$ for $\alpha \in \Phi (P, A)$. Let $\lambda_1 , \cdots , \lambda_\nu$ be a basis of $H_*^\nu$. Set $\gamma_j (\nu) = \lambda_j \circ J_\nu$ for $\text{Re} (\nu, \alpha) > q$. Let $\mu_1 , \cdots , \mu_\nu$ be a basis of $F^\nu$. We apply the earlier observations and Theorem 3.4 to find $p_{ij} \in U(g_\nu)$ (depending on $F$ and $\nu$ but not on $\nu$) such that

$$\sum_{i,j} (\pi_\nu (g) \otimes \mu_\nu (g)) (p_{ij} (\nu) \otimes \mu_j (\nu)) = \xi_j (\nu)$$

$k = 1, \cdots , pr$ defines a basis of $((I_{P,\nu}^* \otimes \kappa |_{P \cap K})$. Fix $\nu_0 \in \alpha_\nu^*$ such that $\text{Re} (\nu_0 , \alpha) > q$ for $\alpha \in \Phi (P, A)$. Let $Z_j = \text{Ker} T_j$. Then the above considerations imply that dim $C \xi_i (\nu_0) |_{Z_j} = pr - p = t$. By relabeling we may assume that $\xi_i (\nu_0) |_{Z_j} = 1, \cdots , t$ are linearly independent in an open neighborhood, $U$, of $\nu_0$. Hence there exist holomorphic functions $a_{ij} , 1 \leq j \leq t, 1 \leq i \leq p$ on $U$ such that

$$\xi_{i+1} (\nu) |_{Z_j} = \sum_{j=1}^t a_{ij} (\nu) \xi_j (\nu).$$

Set $\varphi_i (\nu) = \xi_{i+1} (\nu) - \sum_{j=1}^t a_{ij} (\nu) \xi_j (\nu)$ for $\nu \in U$ and $1 \leq i \leq p$. Then $\varphi_i (\nu) |_{Z_j} = 0$, and $\varphi_1 (\nu) , \cdots , \varphi_p (\nu)$ are linearly independent for $\nu \in U$. Thus $\{ \varphi_i (\nu) \}_{n=1}^p$ is a basis of $\text{Wh}_\gamma ((I_{P,\nu}^* \otimes \kappa |_{P \cap K})$. For $\nu \in U$.

We now use the notation of the previous section. Theorem 5.1 implies that if $\nu \in U$ then there exist $\beta_i (\nu) \in H_*^\nu$ such that if $f \in U_{\nu, \gamma}$ then
\[ \varphi_i(\nu)(f) = \beta_i(\nu) \left( \int_N \eta(n)^{-1} f(w_o n) dn \right). \]

Let \( \nu_i \in H_i \) be such that \( \mu_i(\nu_j) = \delta_{ij} \). Fix \( u \in C_c(\mathbb{N}) \) satisfying

\[ \int_N \eta(n)^{-1} u(n) dn = 1. \]

Set \( f_{x_i}(p w_o n) = a_i(p) u(n) v_i \) for \( p \in P \), \( n \in \mathbb{N} \) and extend \( f_{x_i} \) to \( G \) by 0. Then \( f_{x_i} \in U_{\ast, \ast} \) and \( \nu \rightarrow f_{x_i} \) is holomorphic from \( a_C^\ast \) to \( I^+_\nu, \nu \). We note that

\[ \varphi_i(\nu)(f_{x, x_i}) = \beta_i(\nu)(u_i). \]

Hence \( \beta_i : U \rightarrow H_i^\ast \) is holomorphic. Since \( \beta_1(\nu), \ldots, \beta_r(\nu) \) are linearly independent for \( \nu \in U \), we can find \( b_{x_i} \) holomorphic on \( a_C^\ast \) such that if

\[ \omega_i(\nu) = \sum b_{x_i} \varphi_i(\nu) \]

then

\[ \omega_i(\nu)(f) = \mu_i \left( \int_N \eta(n)^{-1} f(w_o n) dn \right) \]

for \( f \in U_{\ast, \ast, i} \). This implies that if \( \text{Re} (\nu - \lambda, \alpha) > q \) then \( \omega_i(\nu) = \gamma_i(\nu - \lambda) \).

This implies that if \( q_i = \min_{a \in \Phi(P, A)} (\lambda, \alpha) > 0 \) then \( \gamma_i \) has a holomorphic continuation to \( \text{Re} (\nu, \alpha) > q - q_i, \alpha \in \Phi(P, A) \). The Theorem now follows.

**Corollary 7.3.** Let \( P \) and \( \eta \) be given in one of the following ways

1. \( P \) is minimal and \( \eta \) is generic.
2. \( P \) and \( d_P \) are as in Proposition 5.1.

Then the conclusion of Theorem 7.2 is true.

**Proof.** We have already observed that all of the hypotheses of the Theorem are satisfied in these cases.

§ 8. Applications

In this section we give several corollaries to Theorem 7.2 and Corollary 7.3. Let \( G \) be a real reductive group of inner type. We fix the notation as in Section 4. We assume that the Lie algebra of \( \mathfrak{g} \) is as in the case of the BC_n examples of Section 4. Let \( P \) be the parabolic subgroup corresponding to the parabolic subalgebra given as in that section. Let \( \mathfrak{g}_n = \mathfrak{l} \oplus \mathfrak{m} \oplus \mathfrak{u} \). Then \( (\mathfrak{g}_n, \mathfrak{a}_n) \) has a \( C_n \)-root system. Let \( w_o \) be as in Proposition 5.1 for this case.

**Theorem 8.1.** Let \( U = \exp \mathfrak{u} \) and let \( \eta \) be a unitary character of \( U \) such
that $d\eta$ is non-degenerate. Let $\sigma$ be a finite dimensional irreducible representation of $^o M$. If $f \in I_{\tau,\sigma}$ then set

$$J_{\tau,\sigma}(f) = \int_U \eta(u)^{-1} f_{\tau,\sigma}(w_u u) du.$$ 

Then there exists $q_\sigma \geq 0$ such that the above integral converges uniformly in compacta of $\{ \nu \in a_{\xi}^\vee | \Re(\nu,\alpha) > q_\sigma, \alpha \in \Phi(\delta \otimes u, a_\nu) \}$. $J_{\tau,\sigma}$ has a weakly holomorphic extension to $a_{\xi}^\vee$.

**Proof.** Let $G_i$ be the connected subgroup of $G$ with Lie algebra $g_i \oplus G_1$. Set $K_i = K \cap G_i$. If $f \in I_{\tau,\sigma}$ then $S(f) = f|_{K_i} \in I_{K_i,\sigma}$. $S(\pi_{\tau,\sigma}(g)f) = \pi_{\tau,\sigma}(g)S(f)$. The result therefore follows of Corollary 7.3.

We now give an application to a minimal parabolic subgroup. Let $P_\sigma = \theta P_\sigma$, and $N_\sigma = \theta(N_\sigma)$.

**Theorem 8.2.** Let $(\sigma, H_o)$ be an irreducible representation of $^o M_o$ an arbitrary unitary character of $N_o$. Define for $\nu \in (a_\nu)^{\xi}$

$$J_{\eta,\sigma}(f) = \int_{N_o} \eta(n)^{-1} f_{\tau,\sigma}(w_o n) dn.$$ 

Then the integral above converges absolutely and uniformly in $\{(a_\nu)^{\xi} \mid \Re(\nu,\alpha) > 0, \alpha \in \Phi(P_o, A_o)\}$. Furthermore, $J_{\tau,\sigma}$ has a meromorphic continuation to $(a_\nu)^{\xi}$.

**Note.** The proof contains some information on the location of the singularities.

**Proof.** The proof of the first assertion is essentially the same as (1) in Proposition 7.1.

Let $A$ be the set of simple roots of $\Phi(P_o, A_o)$. Let $F = \{ \alpha \in A \mid d\eta|_{a_\alpha} \neq 0 \}$. Let $(P_F, A_F)$ be the corresponding parabolic subgroup (cf. $M_F = P_F \cap \theta(P_F)$ and $^*P_F = P_o \cap M_F, ^*N_F = N_o \cap M_F, ^*N_F = \theta(^*N_F), a_F = a_o \cap ^*m_F$. Then $\alpha_\sigma = \alpha_F \oplus a_F$. If $\nu \in (a_\nu)^{\xi}$ then set $^*\nu_F = \nu|_{a_F}$. If $m \in M_F$ and if $f \in I_{P_F,\sigma}$ then put

$$\lambda_{\nu}(f)(m) = \int_{N_F} f_{\tau,\sigma}(m n) dn.$$ 

This integral converges absolutely for $\Re(\nu, \alpha) > 0$, $\alpha \in \Phi(P_o, A_o)$. $\lambda_{\nu}(\pi_{P_o,\sigma},(m)f) = \pi_{P_F,\sigma},(m)(f)$ for all $m \in M_F$.

(1) There is a holomorphic function $\gamma$ on $(a_F)^{\xi}$ such that if $f \in I_{P_F,\sigma}$ then

$$\nu \longrightarrow \gamma(\nu)\lambda_{\nu}(f)$$
extends to a holomorphic mapping of \((\alpha, \epsilon)\) into \(I_{P_F, \epsilon}^*\).

This is a direct consequence of the main result on in [S]. Let \(s_\circ\) be the longest element relative to \(*P_F\) of the relative to \(A_n\). Let \(w_\circ \in K \cap M_F\) be a representative of \(w_\circ\). Put \(\eta(w_\circ^{-1} n w_\circ)\) for \(n \in N_F\). Then \(\mu\) is a generic character of \(*N_F\). If then (up to normalization of measures)

\[
\int_{*N_F} \eta(\tilde{n})^{-1} \phi_{\omega, \varphi}(\tilde{n}) d\tilde{n} = \int_{*N_F} \mu(n)^{-1} (\pi_\omega(w_\circ^{-1}) \phi)_{\omega, \varphi}(w_\circ, n) dn = J_{*P_F, \omega, \varphi}(\pi_\omega(w_\circ^{-1}) \phi).
\]

Here \(\pi_\omega(k) \varphi(x) = \varphi(kx), k, x \in K \subset P_F\). Up to normalization of invariant measures, one has

\[
J_{*P_F, \omega, \varphi}(f) = J_{*P_F, \omega, \varphi}(\pi_\omega(w_\circ^{-1}) \lambda(f)).
\]

The result now follows from Corollary 7.3.

Appendix. The proof of Lemma 2.1

The proof uses a result on the group algebra of the symmetric group which we do first. Let \(S_n\) denote the symmetric group on \(n\) letters. Let \(\sigma_i\) denote the cycle \((12 \cdots i) \in S_n\) \((\sigma_i = 1)\). We use the notation \(C[S_n]\) for the group algebra of \(S_n\) over \(C\). Set \(\gamma = \Sigma_{i=1}^n \sigma_i \in C[S_n]\).

**Proposition A.1.** \((\gamma - n) \Pi_i=i=1(\gamma - i) = 0\) in \(C[S_n]\).

The proof of this assertion uses several intermediate results.

**Lemma A.2.** If \(u \in S_n\) and if \(u = \sigma_{i_1} \cdots \sigma_{i_t} = \sigma_{j_1} \cdots \sigma_{j_t}\) with \(i_p \geq p\) and \(j_p \geq p\), \(p = 1, \cdots, \) then \(i_p = j_p\) for all \(p = 1, \cdots, r\).

**Proof.** By induction on \(r\). If \(r = 1\) then the result is clear. Our assumptions on the \(i_p\) and the \(j_p\) imply that \(u_i = u_j = r\). We prove this by induction on \(r\). If \(r = 1\) and if \(j \geq 1\) then \(\sigma_j = 1\). Assume for \(r = k - 1\). Then \(\sigma_{i_k} \cdots \sigma_{i_1} i_1 = k - 1\). Since \(i_k \geq k\), \(\sigma_{i_k}(k - 1) = k\). Thus \(i_k = j_k\). Hence \(\sigma_{i_r} \cdots \sigma_{i_2} = \sigma_{j_r} \cdots \sigma_{j_2}\) so \(i_k = j_k\) for \(k = 2, \cdots, r\).

**Corollary A.3.** If \(u \in S_n\) then \(u\) can be written uniquely in the form

\[
u
u
u
u
\]

with \(i_j \geq j\).

**Proof.** The uniqueness follows from Lemma A.2. The number of elements of the form \(\sigma_{i_n} \cdots \sigma_{i_j}\) with \(i_j \geq j\) is \(n!\) by Lemma A.2. Thus every element of \(S_n\) has such an expression.
Corollary A.4. Set \( \mu = \sum w \in \mathfrak{sl}_n \omega \) and \( \gamma_k = \sum_{j \leq k} \sigma_j \). Then

\[
(\gamma - \gamma_{n-1}) \cdots (\gamma - \gamma_2)(\gamma - 1)\gamma = \mu.
\]

Lemma A.5. Let \( u = \sigma_{i_r} \cdots \sigma_{i_1}, \ i_j \geq j \). If \( t \leq r \) then

\[
\sigma_t u = \sigma_{i_r} \cdots \sigma_{i_1}, \ j_p \geq p, \ p = 1, \ldots, r.
\]

Proof. The following formulas are easily checked by direct calculation

(i) \( \sigma \sigma_j = \sigma_j \sigma_{j-1} \) for \( 2 \leq i \leq j \).

(ii) \( \sigma \sigma_j \sigma_{j-1} = 1 \) for \( j = 1, \ldots, r-1 \).

We now prove the Lemma in the case when \( t = 2 \). Suppose that \( i_r > i_{r-1} \) then (ii) implies that \( \sigma \sigma_{i_r} \sigma_{i_{r-1}} = \sigma_{i_{r-1}} \sigma_{i_r} \). So

\[
\sigma_{i_r} \sigma_{i_{r-1}} = \sigma_{i_{r-1}} \sigma_{i_r}.
\]

which implies the Lemma in this case.

If \( i_r = i_{r-1} = p \) then we note that (ii) implies

(iii) \( \sigma \sigma_p = \sigma_p \sigma_{p-1} \).

Hence \( \sigma_{i_r} \sigma_{i_{r-1}} \cdots \sigma_{i_1} \).

Suppose that \( i_r < i_{r-1} \) then (i) implies that

\[
\sigma_{i_r} \sigma_{i_{r-1}} = \sigma_{i_{r-1}} \sigma_{i_r} \sigma_{i_{r-2}} \sigma_{i_1} \cdots \sigma_{i_1}.
\]

So (iii) implies that

\[
\sigma \sigma_{i_r} \sigma_{i_{r-1}} \cdots \sigma_{i_1} = \sigma_{i_{r-1}} \sigma_{i_r} \sigma_{i_{r-2}} \sigma_{i_1} \cdots \sigma_{i_1} = \sigma_{i_{r-1}} \sigma_{i_r} \sigma_{i_{r-2}} \sigma_{i_1} \cdots \sigma_{i_1}.
\]

This completes the proof in the case when \( t = 2 \).

Assume the result for \( t-1 \geq 2 \) we now prove it for \( t \). We note that \( t \leq i_r \). Suppose that \( t < i_r \). Then (i) implies that

\[
\sigma_t u = \sigma_{i_r} \sigma_{i_{r-1}} \sigma_{i_r-1} \cdots \sigma_{i_1}.
\]

The inductive hypothesis implies that this is equal to

\[
\sigma_{i_r} \sigma_{i_{r-1}} \cdots \sigma_{i_j} \quad \text{with} \quad j_p \geq p.
\]

To this we apply (i) with \( j = 2 \) and get

\[
\sigma_t u = \sigma \sigma_{i_r} \sigma_{i_{r-1}} \sigma_{i_r-1} \cdots \sigma_{i_1}.
\]

In this case the inductive step now follows from the case \( t = 2 \).

Suppose now that \( i_r = t \). Then \( t = i_r = r \). So
\[ \sigma_{j_0} \sigma_{j_1} \cdots \sigma_{j_{l-1}} = \sigma_{j_0} \sigma_{j_{l-1}} \cdots \sigma_{j_1} \]

by (ii). The inductive hypothesis now implies that
\[ \sigma_{j_0} \sigma_{j_1} \cdots \sigma_{j_{l-1}} = \sigma_{j_0} \sigma_{j_{l-1}} \cdots \sigma_{j_1} \quad \text{with} \quad j_k \geq k. \]

Thus the inductive step now follows from the case \( t = 2 \).

We are now ready to prove the proposition. Lemma A.5 implies that
\[ \tau_j \prod_{r=0}^{j-1} (\tau - r) = j \prod_{r=0}^{j-1} (\tau - r). \]

Suppose that we have shown that
\[ \prod_{r=0}^{j-1} (\tau - r) = \prod_{r=0}^{j-1} (\tau - r) \]

then the above formula implies
\[ \tau_j \prod_{r=0}^{j-1} (\tau - r) = \tau_j \prod_{r=0}^{j-1} (\tau - r) + j \prod_{r=0}^{j-1} (\tau - r). \]

Since this equation is obvious for \( j = 2 \), Corollary A.4 implies that
\[ \prod_{r=0}^{j-2} (\tau - j) = \mu. \]

The result now follows since \( \tau_j \mu = n \mu. \)

Let now \( \nu_i = (i \cdots n) \) for \( i = 1, \cdots, n \) and \( \tau = \Sigma \nu_i. \)

**Corollary A.6.** \( \tau - n \prod_{j=0}^{n-2} (\tau - j) = 0. \)

**Proof.** Let \( s_n \) be the element of \( S_n \) such that \( s_n i = n + 1 - i. \) Then
\[ \nu_i = \delta_{i}^{-1} s_{n-i}. \]

If we set \( \psi(\Sigma \delta_{i}^{-1} s_{n-i}) = \Sigma \delta_{i}^{-1} s_{n-i} \) then \( \psi \) defines an anti-automorphism of \( C[S_n] \) \( (\psi(xy) = \psi(y)\psi(x)). \) Since \( \tau = \psi(\tau) \), the Corollary follows from Proposition A.1.

We are now ready to prove Lemma 2.1. We return to the pertinent notation in Section 2. Fix \( j \geq 1. \) Define \( V_{p,r} \) to be the space of all \( v \in V \) satisfying the following two conditions

1. \( (Y_i - \psi(Y_i)) \cdots (Y_{p+1} - \psi(Y_{p+1})) v = 0 \) if \( Y_i \in \Sigma_{i \geq j} u_{2i}. \)
2. If \( Y_i \in u_{2r}, \quad r \geq j \) then \( (Y_i - \psi(Y_i)) \cdots (Y_p - \psi(Y_p)) v \in V_r - \Sigma_{r \geq r+1} \).

Here, \( V_q = 0 \) for \( q \leq 0. \) Notice that \( V_{0,r} \subset V_{p+r+1} \), \( V_{0,0} \subset V_{p+1,0} \), \( \bigcup_p V_{p,0} = V \) and \( V_{p+r,0} = V_{r,0} \).

We first observe \( (Q = Q_f) \)

(I) If we can show that \( (Q + 1)^2 \prod_{q=0}^{r-1} (Q + i)^2 V_{p+q} \subset V_{p+r-1} \) then Lemma 2.1 follows.

Indeed, set \( f_{p,r}(T) = (T + 1)^2 \prod_{q=0}^{r-1} (T + q)^2. \) We note that \( V_r \subset V_{0,r} \) and that \( V_{r+1} \cap V_r \subset \bigcup_{q \leq r} V_{r,q}. \) Thus, \( \prod_{q \leq r} f_{r+1-q}(Q) V_r \subset V_r \cap V_{r-1,0}. \) Hence \( \prod_{q \leq r} f_{r+1-q}(Q) V_r \subset V_r \cap V_{r-1,0}, \) etc.. Since \( \delta_0,0 = (0), \) (I) implies the Lemma.

We are left with the proof of (I). If \( x \in u \) then set \( x' = x - \psi(x). \)

Let \( v \in V_{p,r} \) and let \( Y_i \in u_{2r} \) with \( r_i \geq j, i = 1, \cdots, p. \) Then \( QY_1 \cdots Y_p v = 0. \) Hence \( (\dim u_{2j} = m) \)
\[ Y'_1 \cdots Y'_p Q v = \sum_{i=1}^{p} Y'_1 \cdots Y'_{i-1} [Y_i, \mathcal{Q}] Y'_{i+1} \cdots Y'_p v \]

\[ = \sum_{i=1}^{p} \sum_{k=1}^{m} Y'_1 \cdots [Y_{i}, Z_k] X'_k Y'_{i+1} \cdots Y'_p v \]

\[ + \sum_{i=1}^{p} \sum_{k=1}^{m} Y'_1 \cdots Z_k [Y_{i}, X_k] Y'_{i+1} \cdots Y'_p v \]

\[ = \sum_{i=1}^{p} \sum_{k=1}^{m} Y'_1 \cdots [Y_{i}, Z_k] X'_k Y'_{i+1} \cdots Y'_p v \]

\[ + \sum_{i=1}^{p} \sum_{k=1}^{m} \sum_{u=1}^{i-1} Y'_1 \cdots [Y_{u}, Z_k] Y'_{i+1} \cdots Y'_{i-1} [Y_{i}, X_k] Y'_{i+1} \cdots Y'_p v. \]

If \( r_i > j \) then \([Y_{i}, Z_k] \in u_{2,2}(r_i - j)\) so

\[ Y'_1 \cdots [Y_{i}, Z_k] X'_k Y'_{i+1} \cdots Y'_p v \]

\[ = [Y_{i}, Z_k] Y'_1 \cdots X'_k Y'_{i+1} \cdots Y'_p v \]

\[ + \sum_{u=1}^{i-1} Y'_1 \cdots [Y_{u}, [Y_{i}, Z_k]] Y'_{i+1} \cdots Y'_p v \]

which is in \( V_{r - \Sigma_{(r_i-j)-1}} \). Also \( Y'_1 \cdots [Y_{i}, X_k] Y'_{i+1} \cdots Y'_p v = 0 \). Furthermore (as above), if \( r_u > j \) then

\[ Y'_1 \cdots [Y_{u}, [Y_{i}, Z_k]] \cdots Y'_{i-1} X'_k Y'_{i+1} \cdots Y'_p v \in V_{r - \Sigma_{(r_i-j)-1}}. \]

Thus

\[ Y'_1 \cdots Y'_p Q v = \sum_{i=1}^{p} \sum_{k=1}^{m} Y'_1 \cdots [Y_{i}, Z_k] X'_k Y'_{i+1} \cdots Y'_p v \]

\[ + \sum_{i=1}^{p} \sum_{k=1}^{m} \sum_{u=1}^{i-1} \sum_{r_u = k}^{i} Y'_1 \cdots [Y_{u}, Z_k] Y'_{i+1} \cdots Y'_{i-1} [Y_{i}, X_k] Y'_{i+1} \cdots Y'_p v \]

\[ \mod V_{r - \Sigma_{(r_i-j)-1}}. \]

If \( r_i = j \) then \([Y_{i}, Z_k] = -(Y_{i}, X_k) X_0 + u_{i,k} \) with \( u_{i,k} \in u_2 \) and \( \psi(u_{i,k}) = 0 \). As above,

\[ Y'_1 \cdots (u_{i,k} - (Y_{i}, X_k)(X_0 - 1)) X'_k Y'_{i+1} \cdots Y'_p v \in V_{r + \Sigma_{(r_i-j)-1}} \]

as is

\[ Y'_1 \cdots (u_{i,k} - (Y_{i}, X_k)(X_0 - 1)) \cdots Y'_{i-1} [Y_{i}, X_k] Y'_{i+1} \cdots Y'_p v. \]

We therefore have shown that if \( s = \{i | r_i = j\} \) then
(II) \[ Y_1' \cdots Y_p'Qv \equiv -s Y_1' \cdots Y_p'v \]
\[ - \sum_{r_u = k}^{p} \sum_{w \leq i - 1}^{r_u} Y_1' \cdots Y_{u-1}'Y_{u+1}' \cdots Y_{i-1}'[Y_i, Y_u]Y_{i+1}' \cdots Y_p'v \mod V_{r - \Sigma_l(r_l - j - 1)} \]

If \( s = 1 \) and if \( r_u = j \) then this says that \( Y_1' \cdots Y_p'Qv = - Y_1' \cdots Y_{u-1}'Y_u'Y_u'v. \) Thus \( Y_1' \cdots Y_p'(Q + I)Qv = - Y_1' \cdots Y_p'Qv. \) Hence \( Y_1' \cdots Y_p'(Q + I)Qv = 0. \) We thus assume that \( s \geq 2. \) If we write \([Y_i, Y_u] = Y_i'Y_u' - Y_u'Y_i'\) we see that (II) "telescopes" to

(III) \[ Y_1' \cdots Y_p'Qv \equiv - Y_1' \cdots Y_p'v \]
\[ - \sum_{r_u = j}^{p} \sum_{w \leq i} Y_1' \cdots Y_{u-1}'Y_{u+1}' \cdots Y_p'Y_u'v \mod V_{r - \Sigma_l(r_l - j - 1)} \]

(III) implies that, in particular, \( Qu \in V_{p,r}. \) This implies that

\[ Y_1' \cdots Y_p'(Q + I)v \equiv - Y_1' \cdots Y_{u-1}'Y_{u+1}' \cdots Y_p'Y_u'v \mod V_{r - \Sigma_l(r_l - j - 1)} \]

Thus \( Y_1' \cdots Y_p'(Q + I)^2v \) is a sum of terms of the form (up to sign) \( Z_1' \cdots Z_{p-m}'L_1' \cdots L_m'v \) with \( m \geq 2, \) the \( Z_i, L_i \) a relabeling of the \( Y_i \) and \( \L_i \) \( \in \{ u_2 \} \). (III) now implies that

\[ Z_1' \cdots Z_{p-m}'L_1' \cdots L_m'Qv \equiv - \sum_{i=1}^{m} Z_1' \cdots Z_{p-m}'L_{i+1}' \cdots L_{m}'v \mod V_{r - \Sigma_l(r_l - j - 1)} \]

and terms of the same form with larger \( m. \) Thus Corollary A.6 implies that

\[ Z_1' \cdots Z_{p-m}'L_1' \cdots L_m'(Q + ml) \prod_{i=0}^{m-3} (Q + il)v \]

is congruent modulo \( V_{r - \Sigma_l(r_l - j - 1)} \) to a sum of terms of the "ZL" form with larger \( m. \) If we apply this argument at most \( p - 1 \) times we find that

\[ Y_1' \cdots Y_p'(Q + I)^{p-1}(Q + pI) \prod_{i=0}^{p-2} (Q + il)^{p-1}v \]

is congruent modulo \( V_{r - \Sigma_l(r_l - j - 1)} \) to terms of the form

\[ Z_1'Z_2' \cdots Z_{p-s}'L_1' \cdots L_s'v. \]

With \( Z_i \in u_{2q_t}, q_t > j \) for \( i = 1, \ldots, p-s \) and \( L_i \in u_{2j} \) for \( i = 1, \ldots, s. \) The \( Z_i \) and the \( L_i \) constituting (up to sign) a rearrangement of the \( Y_i. \) If we now apply (III) again we find that
\[ Z'_1 Z'_2 \cdots Z'_{p-s} L'_1 \cdots L'_q v \]
\[ \equiv - \sum_{i=1}^{s} Z'_i Z'_2 \cdots Z'_{p-s} L'_{i+1} \cdots L'_{q+1} v \mod V_{r-\Sigma(i+1)} \]

We apply Corollary A.6 again to find that

\[ Y'_1 \cdots Y'_p (Q + I)^{q} \prod_{\kappa=0}^{q} (Q + uI)^{\kappa} v \in V_{r-\Sigma(i+1)} \]

This implies that \((Q + I)^{q} \prod_{\kappa=0}^{q} (Q + uI)^{\kappa} V_{p,r} \in V_{p,r-1}\). This is the assertion in (I). So we are done.

References


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