Limit Multiplicities in $L^2(\Gamma\backslash G)$

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Introduction.

Let G be a linear semi-simple Lie group over $\mathbf{R}$. Let Γ be a discrete cocompact subgroup of G. In our paper $[DeG - W]$, DeGeorge and I observed that a fairly simple formula for the Hilbert-Schmidt norm of certain convolution operators on $L^2(\Gamma\backslash G)$ could be a powerful tool in the analysis of the distribution of multiplicities in $L^2(\Gamma_j\backslash G)$ for $\Gamma_j \supset \Gamma_{j+1}$, Γ_j normal of finite index in Γ and $\cap\Gamma_j = \{1\}$. The sharpest results in $[DeG - W]$, [DeG-W2] were for general discrete series, principal series and general representations of real rank one groups. These theorems were subsequently generalized by several authors notably Miatello [M] (who studied discrete series for non-linear groups using the trace formula with a character) and Delorme [D] (who gave the arbitrary rank generalization of our rank one result).

If $\Gamma\backslash G$ is not compact then the method of $[DeG - W]$, which uses the injectivity radius of $\Gamma\backslash G/K$, cannot be applied. Using a completely different method, Rohlfs and Speh ($[R - S]$) derived a limit formula for the sum of the multiplicities in the cuspidal spectrum of a fixed discrete series L-packet for Γ arithmetic. Savin ($[S]$) observed that in the cocompact case one can modify the argument in $[DeG - W]$ so that the injectivity radius only implicitly plays a role. Using a sequence of ingenious arguments he was able to prove an upper bound for the "limit" multiplicity in the cuspidal spectrum of discrete series in the case when the Γ_j are congruence subgroups. This upper bound fits perfectly with the theorem of $[R-S]$ and the two results combined imply the exact limit multiplicity formula for discrete series.

In this article we return to the original method of $[DeG - W]$ (as it was given in the preprint form of that paper not as in the reprint form). We show that Savin's method can be used to prove an estimate for the Hilbert-Schmidt norm of certain convolution operators on the cuspidal spectrum for arithmetic groups. This is the content of Proposition 5.1 which is the main new result of this paper (and should be considered to be a sharpening of a theorem of Langlands $[L]$, cf. Lemma 3.1.). Using this result we give a proof of Savin's result for Γ arithmetic and a class of Γ_j. In the last section of this paper we show how the Hilbert-Schmidt inequality can be used to prove that the sum of the multiplicities of spherical complementary series is negligible relative to the volume. In other words a "limiting form" of Selberg's conjecture for the "exceptional eigenvalues" is true.

This article is in part expository and in part a research paper. This format allows detailed discussions of some known results. However, the expository portion is by no means encyclopedic and we apologize to those authors whose related results have not been discussed.

It is my pleasure to thank Labesse and Schwermer for running the pleasant and stimulating conference at Luminy on which this volume is based and for accepting a manuscript that is neither expository nor straight research.

1. Some remarks on Hilbert-Schmidt class operators.

If H_1 and H_2 are (separable) Hilbert spaces then a bounded operator $T : H_1 \to H_2$ is said to be of Hilbert-Schmidt class if there exists an orthonormal basis $\{e_n\}$ of H_1 such that

$$\sum \|Te_n\|^2 < \infty.$$

It is standard that if T is of Hilbert-Schmidt class and if $\{f_n\}$ is another orthonormal basis of H_1 then

$$\sum \|Tf_n\|^2 = \sum \|Te_n\|^2.$$

The common value is denoted $\|T\|_{HS}^2$. It is also standard that T is of Hilbert-Schmidt class if and only if T^* is. We set $HS(H_1, H_2)$ equal to the space of all Hilbert-Schmidt operators from H_1 to H_2. Then $HS(H_1, H_2)$ is the completion of the space of finite rank operators with respect to the norm $\|...\|_{HS}$.

Let $(X, \mathcal{A}, \mu)$ be a measure space with X a set, $\mathcal{A}$ a σ-algebra of subsets of X and μ a (positive) measure on $\mathcal{A}$. We assume that there exists a countable σ-subalgebra, $\mathcal{B}$, of $\mathcal{A}$ such that if $A \in \mathcal{A}$ then there exists $B \in \mathcal{B}$ such that $\mu(A - A \cap B) = \mu(B - A \cap B) = 0$. We also assume that $X = \cup B_j$ (countable union) with $B_j \in \mathcal{B}$ and $\mu(B_j) < \infty$ for all j. Then $L^2(X, \mu)$ is separable.

Let $V \subset L^2(X, \mu)$ be a closed subspace. And let T be a bounded linear map from V to $L^2(X, \mu)$. If $\{e_n\}$ is an orthonormal basis of V then we set

$$B_T(x) = \sum |(Te_n)(x)|^2.$$

Then B_T is defined μ-almost everywhere (with a non-negative real value or with the value ∞) and is μ-measurable. The following simple lemma will be used several times in this exposition.

Lemma 1.1. *Suppose that γ is a μ-measurable function on X such that for each $f \in V$*

$$|Tf(x)| \leq \gamma(x)\|f\|_2 \ \mu - a.e.$$

Then

(1) $B_T(x) \leq \gamma(x) \ \mu - a.e.$
(2) *If $\gamma \in L^2(X, \mu)$ then $T \in HS(V, L^2(X, \mu))$ and $B_T \in L^1(X, \mu)$. Furthermore,*

$$\int_X B_T(x) d\mu(x) = \|T\|_{HS}^2 .$$

Proof. Let Ω be a countable dense subset of $\mathbf{C}$ containing 0 and 1 and let $\{e_n\}$ be an orthonormal basis of V. Let W be the set of functions on X of the form $\sum a_j e_j$ with $a_j \in \Omega$ and all but a finite number of the $a_j = 0$. Let X_1 be a subset of X such that

$\mu(X - X_1) = 0$ and such that $\gamma(x)$ and $Tf(x)$ are defined and finite for all $f \in W$ and $x \in X_1$. We also assume that

$$|Tf(x)| \leq \gamma(x)\|f\|_2 \text{ for } x \in X_1.$$

Let $N < \infty$. If $\sum_{j \leq N} a_j e_j \in W$ then

$$|\sum_{j\leq N} T(a_j e_j)(x)| = |\sum_{j\leq N} a_j(Te_j)(x)| \leq$$
$$(\sum_{j\leq N} |a_j|^2)^{1/2}(\sum_{j\leq N} |Te_j(x)|^2)^{1/2}$$

for all $x \in X_1$. Since Ω is dense in $\mathbf{C}$ the above inequality is true for all N and all $a_j \in \mathbf{C}$ for $x \in X_1$. This implies that if $f \in V_N = \sum_{j \leq N} \mathbf{C}e_j$ then

$$|Tf(x)| \leq \|f\|_2(\sum_{j\leq N} |Te_j(x)|^2)^{1/2},\ x \in X_1.$$

Hence,

$$|Tf(x)| \leq \|f\|_2 B_T(x)^{1/2},\ x \in X_1,\ f \in V.$$

We now apply the hypothesis on γ. If we argue as above we find that

$$|\sum_{j\leq N} T(a_j e_j)(x)| \leq \gamma(x)(\sum_{j\leq N} |a_j|^2)^{1/2}$$

for all $a_j \in \mathbf{C}$ and $x \in X_1$. If we apply this to a_j equal to the complex conjugate of $Te_j(x)$ for $x \in X_1$ then we find that

$$\sum_{j\leq N} |(Te_j)(x)|^2 \leq \gamma(x)(\sum_{j\leq N} |(Te_j)(x)|^2)^{1/2}.$$

This implies that if $x \in X_1$ then

$$\sum_{j\leq N} |(Te_j)(x)|^2 \leq \gamma(x)^2.$$

Since N is arbitrary, $B_T(x) \leq \gamma(x)^{1/2}$ for $x \in X_1$. Now it is clear that

$$\|T\|_{HS}^2 = \int_X B_T(x)d\mu(x) \leq \int_X \gamma(x)^2 d\mu(x).$$

This completes the proof of the lemma.

Corollary 1.2. *If T is the identity map of V then set $B_V = B_T$. We have*

$$\dim\ V = \int_X B_V(x)d\mu(x).$$

Proof. $\int_X B_T(x)d\mu(x) < \infty$ if and only if T is Hilbert-Schmidt. That is, if and only if $\dim\ V < \infty$. If $\dim\ V < \infty$ then $\dim\ V = \|T\|_{HS}^2$.

2. Limit multiplicities in the cocompact case.

The purpose of this section is to give an exposition of the original argument of $[DeG-W]$ (essentially as it appeared in the preprint of the paper not as it appears in the reprint). We will also give a variant of the argument due to Savin that gives another proof of the key proposition and is one essential ingredient in his work in the case of congruence subgroups. In our proof of Savin's result we will use a mixture of both techniques.

Let G be a linear connected semi-simple Lie group over $\mathbf{R}$. Fix dg an invariant measure on G (we will normalize it shortly). Let K be a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote respectively the Lie algebras of G and K. Let θ be the Cartan involution of $\mathfrak{g}$ with respect to K. Set $\mathfrak{p} = \{X \in \mathfrak{g} | \theta X = -X\}$. We denote by $\langle\ ,\ \rangle$ the inner product on $\mathfrak{p}$ given by the restriction of the Killing form. We identify $T(G/K)_{1K}$ with $\mathfrak{p}$ (as usual) and also denote by $\langle\ ,\ \rangle$ the G-invariant Riemannian structure on G/K corresponding to $\langle\ ,\ \rangle$ on $\mathfrak{p}$. We set $\sigma(x) = d(xK, 1K)$, where d is the Riemannian distance on G/K. Then $\sigma(\exp X \cdot k) = \langle X, X\rangle^{1/2}$ for $X \in \mathfrak{p}$ and $k \in K$. We have

(1) $\sigma(x) = \sigma(x^{-1})$.

(2) $\sigma(xy) \leq \sigma(x) + \sigma(y)$.

Let Γ be a cocompact, torsion free, discrete subgroup of G. Then $\langle\ ,\ \rangle$ pushes down to a Riemannian structure on the compact manifold $\Gamma\backslash G/K$. Let r_Γ denote the injectivity radius of $\Gamma\backslash G/K$. We can describe r_Γ in our context as follows: let for $r > 0$,

$$B_r = \{x \in G | \sigma(x) < r\}.$$

Then $2r_\Gamma$ is the maximum of the r such that

$$\{g\gamma g^{-1} | g \in G, \gamma \in \Gamma\} \cap B_r = \{1\}.$$

If $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ with Γ_i normal and of finite index in Γ then $\{\Gamma_i\}$ is called a tower of subgroups of Γ.

Lemma 2.1 (cf. $[DeG-W]$). *If Γ is a cocompact discrete subgroup of G then a tower exists for Γ. If $\{\Gamma_j\}$ is a tower for Γ then*

$$\lim_{j\to\infty} r_{\Gamma_j} = \infty.$$

If Γ is arithmetic then a tower for Γ can be gotten by taking Γ_j to be the congruence subgroup of Γ of level j. In the general case the existence of towers can be deduced from [$B1$;Proposition 2.3 (2)]. The second assertion of Lemma 2.1 can be proved as follows.

If $\gamma \in \Gamma$ let $[\gamma]$ denote its Γ-conjugacy class. Set $\ell([\gamma]) = \inf\{\sigma(g\gamma g^{-1})|g \in G\}$. Let $0 < r < \infty$ be given. We assert that the number of $[\gamma]$ such that $\ell([\gamma]) \leq r$ is finite. Indeed, let $S = \{[\gamma]|\ell([\gamma]) \leq r\}$. Let F be a compact fundamental domain for Γ. If $\mu \in S$ then there exists $\gamma_\mu \in \mu$ and $g_\mu \in F$ such that $\sigma(g_\mu\gamma_\mu g_\mu^{-1}) \leq r$. If $g \in F$ then $\sigma(g) \leq C < \infty$. Thus $\sigma(\gamma_\mu) \leq r + 2C$ for $\mu \in S$. Since Γ is discrete, this implies that S is finite. Each Γ_j is assumed to be normal in Γ and since $\cap\Gamma_j = \{1\}$. It now follows that $\lim_{j\to\infty} r_{\Gamma_j} = \infty$.

Fix Γ, cocompact, torsion free, discrete subgroup of G. Let π_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$.

Theorem 2.2. $[DeG - W]$ *If* $\Gamma \subset G$ *is as above and if* $u \in L^2(G)$, supp $u \subset B_{r_\Gamma}$ *then*

$$\|\pi_\Gamma(u)\|_{HS}^2 = vol(\Gamma\backslash G)\|u\|_2^2 .$$

In $[DeG - W]$ we used the trace formula to prove the above equation. However, in the original preprint of the paper a proof of the above formula with " = " replaced by " $\leq$ " was given using the methods of section 1. We will now give this argument with a slight sharpening to prove equality.

Let p_Γ be the canonical projection of G onto $\Gamma\backslash G$.

(1) If $x \in G$ and if $v, w \in B_{r_\Gamma}$ and $p_\Gamma(xv) = p_\Gamma(xw)$ then $v = w$.

Indeed, then there exists $\gamma \in \Gamma$ such that $\gamma xv = xw$. Thus $x^{-1}\gamma x = wv^{-1}$. But $wv^{-1} \in B_{2r_\Gamma}$. Thus $x^{-1}\gamma x = 1$.

Now let u be as in the statement of the theorem. Let $\varphi \in L^2(\Gamma\backslash G)$. Then

$$\pi_\Gamma(u)\varphi(x) = \int_G \varphi(xg)u(g)dg = \int_G \varphi(g)u(x^{-1}g)dg =$$

$$\int_{xB_{r_\Gamma}} \varphi(g)u(x^{-1}g)dg.$$

Thus

$$|\pi_\Gamma(u)\varphi(x)| \leq (\int_{xB_{r_\Gamma}} |\varphi(g)|^2 dg)^{1/2}(\int_{xB_{r_\Gamma}} |u(x^{-1}g)|^2 dg)^{1/2}.$$

Now (1) above implies that

$$|\pi_\Gamma(u)\varphi(x)| \leq \|\varphi\|_2\|u\|_2.$$

Note the two meanings of $\|...\|_2$. Lemma 1.1 implies that if $T = \pi_\Gamma(u)$ then $B(x) = B_T(x) \leq \|u\|_2$ *a.e.*

To prove the reverse inequality we define for $g \in G$, $\tilde{u}_g \in L^2(\Gamma\backslash G)$ as follows. If $x \in gB_{r_\Gamma}$ then $\tilde{u}_g(p_\Gamma(x)) = \bar{u}(g^{-1}x)$ if $x \notin p_\Gamma(gB_{r_\Gamma})$ then $\tilde{u}_g(x) = 0$. Then

$$\pi_\Gamma(u)\tilde{u}_g(g) = \int_G \tilde{u}_g(gx)u(x)dx =$$

$$\int_{B_{r_\Gamma}} |u(x)|^2 dx = \|\tilde{u}_g\|_2\|u\|_2.$$

Thus Lemma 1.1 implies that $\|\tilde{u}_g\|_2\|u\|_2 \leq B(g)\|\tilde{u}_g\|_2$ for *a.e.* g. Since $\|\tilde{u}_g\|_2 = \|u\|_2$. This implies that $\|u\|_2 \leq B(g)$ *a.e.* The result now follows from Lemma 1.1.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. Fix $\omega_o \in \hat{G}$ and $(\pi, H) \in \omega_o$. Let $v \in H$ be a K-finite unit vector. Let χ_r denote the characteristic function of B_r. We put $\varphi_r(g) = \chi_r(g)\langle v, \pi(g)v\rangle$ set $u_\Gamma = \varphi_{r_\Gamma}$. Then from the definition of the Hilbert-Schmidt norm we have

$$\|\pi(\varphi_r)\|_{HS}^2 \geq |\langle \pi(\varphi_r)v, v\rangle|^2 = (\int_{B_r} |\langle \pi(g)v, v\rangle|^2 dg)^2 = \|\varphi_r\|_2^4 .$$

On the other hand, $L^2(\Gamma\backslash G)$ is as a representation of G, $\pi_\Gamma = \oplus_{\omega\in\hat{G}} N_\Gamma(\omega)\omega$, with $N_\Gamma(\omega) < \infty$. Hence

$$\|\pi_\Gamma(u_\Gamma)\|_{HS}^2 \geq N_\Gamma(\omega_o)\|\pi(u_\Gamma)\|_{HS}^2 \geq N_\Gamma(\omega_o)\|u_\Gamma\|_2^4 .$$

Theorem 2.1 now implies

$$N_\Gamma(\omega_o)\|u_\Gamma\|_2^4 \leq vol(\Gamma\backslash G)\|u_\Gamma\|_2^2 .$$

We have proved

Lemma 2.3. $N_\Gamma(\omega_o) \leq vol(\Gamma\backslash G)/\|u_\Gamma\|_2^2$.

Let $\hat{G}_d$ denote the set of equivalence classes of irreducible square integrable representations of G. If $\omega \in \hat{G}_d$ let $d(\omega)$ denote its formal degree.

Proposition 2.4. *Let* $\{\Gamma_j\}$ *be a tower for* Γ.

(1) *If* $\omega_o \notin \hat{G}_d$ *then* $\lim_{j\to\infty} N_{\Gamma_j}(\omega_o)/vol(\Gamma_j\backslash G) = 0$.

(2) *If* $\omega_o \in \hat{G}_d$ *then* $\limsup_{j\to\infty} N_{\Gamma_j}(\omega_o)/vol(\Gamma_j\backslash G) \leq d(\omega_o)$.

Proof. If $\omega_o \notin \hat{G}_d$ set $d(\omega_o) = 0$. Since $\lim_{j\to\infty} \int_{B_{r_{\Gamma_j}}} |\langle \pi(g)v, v\rangle|^2 dg = d(\omega_o)^{-1}$. (1) and (2) now follow from Lemma 2.3.

At this point we give the argument of Savin that also proves the above proposition. Fix $\tau \in \hat{K}$ such that the multiplicity of τ in H is 1 (such a τ exists by the lowest K-type theorem of [V].) Let $v \in H$ be a unit vector such that relative to a choice of a maximal torus, T, of K and a system of positive roots for (K, T), v is a highest weight vector. Let $W_j = \{Tv | T \in Hom_G(H, L^2(\Gamma_j\backslash G))\}$. Note that $\dim W_j = N_{\Gamma_j}(\omega_o)$. Let $B_j = B_{W_j}$

(see Corollary 1.2). We note that since Γ_j is normal in Γ, Γ acts unitarily on the left on $L^2(\Gamma_j\backslash G)$ and the action of Γ preserves W_j. Hence B_j (as a function on G) is left Γ-invariant. Corollary 1.2 implies that

(i) dim $W_j = (\Gamma : \Gamma_j) \int_{\Gamma\backslash G} B_j(x)dx$.

Here $(A : B)$ denotes the index of the subgroup B in the group A. Let $\varphi_r(g) = \chi_{B_r}(g)\langle v, \pi(g)v\rangle$.

Lemma 2.5.

(1) $\pi(\varphi_r)H \subset \mathbf{C}v$.

(2) *Define* λ_r *by* $\pi(\varphi_r)v = \lambda_r v$. *Then*

$$|\lambda_r| = \|\pi(\varphi_r)\|_{HS} \geq \|\varphi_r\|_2^2.$$

Proof. The Shur orthogonality relations imply that the image of $\pi(\varphi_r)$ is contained in the highest weight space of the τ isotypic component of H. Since this describes $\mathbf{C}v$(1) is clear as is the first equation of (2). The lower bound in (2) was proved in the course of the proof of Proposition 2.3.

We now begin the second proof of Proposition 2.4. Let $f \in W_j$ then

$$\lambda_r f(x) = \int_G f(xy)\varphi_r(y)dy = \int_G f(y)\varphi_r(x^{-1}y)dy$$

$$= \int_{\Gamma_j\backslash G} f(y) \sum_{\gamma\in\Gamma_j} \varphi_r(x^{-1}\gamma y)dy.$$

The Schwarz inequality implies that

(*)

$$|\lambda_r f(x)| \leq \|f\|_2 (\int_{\Gamma_j\backslash G} |\sum_{\gamma\in\Gamma_j} \varphi_r(x^{-1}\gamma y)|^2 dy)^{1/2}.$$

If r is fixed and j is sufficiently large ($r_{\Gamma_j} \geq r$) then the sum in the right hand side of the above inequality has only one term corresponding to $\gamma = 1$. We therefore have

$$|f(x)| \leq \|f\|_2 / \|\varphi_r\|_2.$$

Thus if $r_{\Gamma_j} \geq r$ then $B_j(x) \leq \|\varphi_r\|_2^{-2}$. Hence

$$\dim\ W_j / vol(\Gamma_j\backslash G) \leq 1/\|\varphi_r\|_2^2\ .$$

By taking j to infinity and then r to infinity Proposition 2.4 follows.

Although the two arguments are quite similar, it should be pointed out that the second argument uses the existence of a multiplicity one K-type. The first is therefore slightly more elementary. Although the second argument does not work in the non-cocompact case, we shall see that if $L^2(\Gamma_j\backslash G)$ is replaced by its cuspidal part then a detailed analysis of (*) above at the cusps will lead to an estimate so that in the limits the cusps to not contribute. To carry this out it is necessary to use an idea of Langlands which will be described in the next section. In this section we continue with the exposition of the limit multiplicity formula in the cocompact case. We will follow the method in $[DeG-W]$ using the Dirac operator. In Savin's method the Dirac operator is replaced by $d+d^*$ from even forms to odd forms. This latter operator has been analyzed in the noncompact case in $[R-S]$ and their results will be explained in due course.

Fix a maximal torus, T, in K and a system of positive roots, P_k, for the root system of K with respect to T. Let ρ_k be the half sum of the elements of P_k (as a linear functional on $Lie(T)$). Let $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ and let G_u be the connected subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_u$. The map $\varphi(k) = Ad(k)_{|\mathfrak{p}}$ maps K into $SO(\mathfrak{p})$. By going to a two fold covering, $\tilde{K}$ (if necessary), we can lift φ to $\tilde{\varphi} : \tilde{K} \to Spin(\mathfrak{p})$. Let p be the covering homomorphism of $\tilde{K}$ onto K. We assume that $\dim(G/K)$ is even. Then we may choose $(s_\pm, S^\pm)$, half spin representations of Spin(p) and set $\sigma_\pm = s_\pm \circ \tilde{\varphi}$. Let (τ, E) be an irreducible unitary representation of $\tilde{K}$ such that Ker $\tau \otimes \sigma_\pm \supset \ker p$. Then $\tau \otimes \sigma_\pm$ is a representation of K. We can thus form the following vector bundles:

$$E_\tau^\pm = G_u \times_K E \otimes S^\pm.$$

$$E_{\tau,\Gamma}^\pm = \Gamma\backslash G \times_K E \otimes S^\pm$$

over G_u/K and $\Gamma\backslash G/K$ respectively. We can form the corresponding Dirac operators

$$D_\tau : \Gamma^\infty(E_\tau^+) \to \Gamma^\infty(E_\tau^-)\ ,$$

$$D_{\tau,\Gamma} : \Gamma^\infty(E_{\tau,\Gamma}^+) \to \Gamma^\infty(E_{\tau,\Gamma}^-)$$

respectively.

We now normalize the Haar measure on G. Let $n = \dim G/K$ and let $\omega \in \wedge^n \mathfrak{g}_u^*$ be such that integration against the left invariant form corresponding to ω is Haar measure on the simply connected covering group of G_u with total volume 1. If we look upon ω as an element *of* $\wedge^n\ \mathfrak{g}_{\mathbb{C}}^*$ then the restriction of ω to $\mathfrak{g}$ is either ω_o or $i\omega_o$ with $\omega_o \in \wedge^n\mathfrak{g}$. We use ω_o to define dg.

Theorem 2.6. ($[C-G-W]$).*Ind* $D_{\tau,\Gamma} = vol(\Gamma\backslash G) Ind\ D_\tau$.

The index of D_τ can be computed very simply using the Peter-Weyl theorem (and Frobenius reciprocity) the details are in $[DeG-W]$. The result is as follows:

Lemma 2.7.
(1) *If T is not a maximal torus of G_u then Ind $D_\tau = 0$.*
(2) *If T is a maximal torus of G_u and if $\lambda + \rho_k$ is not (G_u, T)-regular then Ind $D_\tau = 0$.*

(3) *If T is a maximal torus of G_u and if $\lambda + \rho_k$ is regular let P be the system of positive roots for (G_u, T) such that $\lambda + \rho_k$ is dominant. Let ρ be the half sum of the elements of P. Let $\epsilon_\tau = 1$ if the irreducible representation of $\tilde{K}$ with highest weight $\rho - \rho_k$ appears in σ_+ and $\epsilon_\tau = -1$ otherwise. Then*

$$\text{Ind } D_\tau = \epsilon_\tau (-1)^{n/2} \prod_{\alpha \in P} \frac{(\lambda+\rho_k,\alpha)}{(\rho,\alpha)} .$$

Combining these two results we have

Corollary 2.8. *If T is a maximal torus of G_u and if $\lambda_\tau + \rho_k$ is (G_u, T)-regular then*

$$\text{Ind } D_{\tau,\Gamma} = \epsilon_\tau \prod_{\alpha \in P} \frac{(\lambda+\rho_k,\alpha)}{(\rho,\alpha)} vol(\Gamma \backslash G)$$

otherwise Ind $D_{\tau,\Gamma} = 0$.

We now compute the index in a second way. We assume that T is a maximal torus of G_u and that $\lambda_\tau + \rho_k$ is regular.

Lemma 2.9. *Let V be a unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character with Harish-Chandra parameter Λ. If the Casimir operator acts on V by $(\lambda_\tau+\rho_k, \lambda_\tau+\rho_k)-(\rho,\rho)$ and if*

$$Hom_K(E_\tau \otimes (S^+ + S^-), V) \neq 0$$

then up to the action of the Weyl group of (G_u, T), $\Lambda = \lambda_\tau + \rho_k$.

Proof. Let P be as above. Let F be the irreducible finite dimensional $(\mathfrak{g}, K)$-module with highest weight $\lambda_\tau + \rho_k - \rho$. Then $E_\tau \otimes (S^+ + S^-)$ is a K-submodule of $F \otimes \wedge \mathfrak{p}_{\mathbf{C}}$. The result now follows from (cf.) $[B-W, II.3.1, I.5.3]$.

Let for $\omega \in \hat{G}$, χ_ω be its infinitesimal character. Let T be as above. Let $\mathfrak{h} = Lie(T)_{\mathbf{C}}$. If $\Lambda \in \mathfrak{h}^*$ let χ_Λ be the infinitesimal character given by the Harish-Chandra isomorphism. Let $\hat{G}_\Lambda = \{\omega \in \hat{G} | \chi_\omega = \chi_\Lambda\}$. Then $\hat{G}_\Lambda$ is a finite set.

Lemma 2.10. *$Ind(D_{\tau,\Gamma}) =$*

$$\sum_{\omega \in \hat{G}_{\lambda_\tau+\rho_\tau}} (\dim Hom_K(E_\tau \otimes S^+, H_\omega) - \dim Hom_K(E_\tau \otimes S^-, H_\omega)) N_\Gamma(\omega).$$

This is proved by a variant of the proof of Matsushima's formula (cf. $[B-W; VII.3.2]$).

If we combine the above results with Proposition 2.4 we have

Theorem 2.11. *Set $S_\tau = \hat{G}_{\lambda_\tau+\rho_k} \cap \hat{G}_d$. Then*

$$\lim_{j\to\infty} \sum_{\omega \in S_\tau} (\dim Hom_K(E_\tau \otimes S^+, H_\omega) - \dim Hom_K(E_\tau \otimes S^-, H_\omega)) N_\Gamma(\omega)/vol(\Gamma_j \backslash G)$$

$$= \epsilon_\tau \prod_{\alpha \in P} \frac{(\lambda + \rho_k, \alpha)}{(\rho, \alpha)} .$$

Note. This result implies that under the above conditions $S_\tau \neq \emptyset$. It therefore gives a proof of the existence of irreducible square integrable representations if T is a maximal torus of G_u. In fact, one can derive a large portion of the theory of the discrete series from the above theorem ($cf.[DeG - W]$).

If we apply the theory of the discrete series ($[H - S]$) then under the above hypothesis there is a unique element, $\omega_{\lambda_\tau+\rho_k} \in S_\tau$ such that $q_\tau(\omega) = \dim Hom_K(E_\tau \otimes S^+, H_\omega) - \dim Hom_K(E_\tau \otimes S^-, H_\omega) \neq 0$. Furthermore, $q_\tau(\omega_{\lambda_\tau+\rho_k}) = \epsilon_\tau$. We note that Harish-Chandra's parametrization of the discrete series ($[H]$) implies that the $\omega_{\lambda_\tau+\rho_k}$ exhaust the $\hat{G}_d$. If in addition we have $\omega = \omega_{\lambda_\tau+\rho_k}$ integrable then Langlands has shown that $N_\Gamma(\omega) = vol(\Gamma\backslash G)d(\omega)$. Since $d(\omega_\Lambda) = c\prod_{\alpha\in P}|(\Lambda,\alpha)|$ with c only depending on the choice of Haar measure, the above result combined with these observations imply

Theorem 2.12. $[DeG - W]$.

(1) *If $\omega \in \hat{G}_d$ has infinitesimal character χ_Λ then*

$$d(\omega) = \prod_{\alpha\in P} \frac{|(\Lambda,\alpha)|}{(\rho,\alpha)}$$

with the above normalization of Haar measure.

(2) *If $\omega \in \hat{G}_d$ and if $\{\Gamma_j\}$ is a tower for the cocompact discrete subgroup Γ of G then*

$$\lim_{j\to\infty} N_{\Gamma_j}(\omega)/vol(\Gamma_j\backslash G) = d(\omega).$$

3. Langlands proof of the traceability of the cuspidal spectrum.

Although the main theorem of this section will not be used in a significant way in the rest of this article, we include an exposition of it since it contains the second essential idea in Savin's proof of his limit multiplicity formula. Let G be an open subgroup of the real points of a reductive algebraic group defined over $\mathbf{Q}$. Let Γ be an arithmetic subgroup of G. (More generally we may assume that G is a real reductive group and that Γ satisfies Langlands' axioms $[L]$.) Let K be a maximal compact subgroup of G. Fix an invariant measure dg on G and push it down to $\Gamma\backslash G$. Let π_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$. If $f \in L^1(G)$, $\varphi \in L^2(\Gamma\backslash G)$ then let $\pi_\Gamma(f)\varphi$ be defined (as usual) by

$$\langle\pi_\Gamma(f)\varphi,\eta\rangle = \int_G f(g)\langle\pi_\Gamma(g)\varphi,\eta\rangle dg.$$

If $C(\Gamma\backslash G)$ is the space of all continuous functions on $\Gamma\backslash G$ then $\pi_\Gamma(f)L^2(\Gamma\backslash G) \subset C(\Gamma\backslash G)\cap L^2(\Gamma\backslash G)$ and if $\varphi \in C(\Gamma\backslash G)\cap L^2(\Gamma\backslash G)$ then

$$\pi_\Gamma(f)\varphi(x) = \int_G \varphi(xg)f(g)dg.$$

If (P,A) is a cuspidal parabolic (*i.e.* P is the group of real points of a $\mathbf{Q}$-parabolic and A is the group of real points of a $\mathbf{Q}$-split component), $P = MAN$ a Langlands decomposition (over $\mathbf{Q}$) and if $\varphi \in C(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)$ then we set

$$\varphi_P(g) = \int_{\Gamma\cap N\backslash N} \varphi(ng)dn.$$

Here, we note that $\Gamma \cap N\backslash N$ is compact and choose the invariant measure on N, dn, such that the total measure of $\Gamma \cap N\backslash N$ is 1. Let ${}^oL^2(\Gamma\backslash G)$ denote the closure in $L^2(\Gamma\backslash G)$ of the space of $\varphi \in C(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)$ such that $\varphi_P = 0$ for all proper cuspidal parabolic subgroups. Notice that if $f \in L^1G)$ then $\pi_\Gamma(f)$ preserves ${}^oL^2(\Gamma\backslash G)$.

We assume that the split component of G is $\{1\}$.

Lemma 3.1 ([*L*]). *Let $f \in C_c^1(G)$(i.e. f is compactly supported and has one continuous derivative). Then there exists a continuous $\gamma \in L^2(\Gamma\backslash G)$ such that*

$$|\pi_\Gamma(f)\varphi(x)| \leq \gamma(x)\|\varphi\|_2$$

for all $\varphi \in^o L^2(\Gamma\backslash G)$.

Proof. Let (P,A) be a minimal cuspidal pair. Let Δ be the set of simple roots of (P,A), $\Delta = \{\alpha_1, ..., \alpha_\ell\}$. Let (P_i, A_i) be the maximal cuspidal pair corresponding to α_i. That is, $A_i = \{a \in A | a^{\alpha_j} = 1\ ,\ j \neq i\}$. Then $\dim A_j = 1$. Let $\varphi \in C(\Gamma\backslash G) \cap^o L^2(\Gamma\backslash G)$. Then

(1) $\int_{\Gamma\cap N_i\backslash N_i} \varphi(ng)dg = 0$.

Since N_i is normal in N, $N_i(\Gamma \cap N)$ is a Lie subgroup of N. We calculate

$$\pi_\Gamma(f)\varphi(g) = \int_G \varphi(gx)f(x)dx = \int_G \varphi(x)f(g^{-1}x)dx =$$

$$\int_{N_i(\Gamma\cap N)\backslash G}\int_{N_i(\Gamma\cap N)} \varphi(ux)f(g^{-1}ux)\mathrm{dudx} \ =$$

$$\int_{N_i(\Gamma\cap N)\backslash G}\left\{\sum_{\delta\in\Gamma\cap N_i\backslash\Gamma\cap N}\int_{\Gamma\cap N_i\backslash N_i}\varphi(n\delta x)\sum_{\zeta\in\Gamma\cap N_i} f(g^{-1}\zeta n\delta x)dn\right\}dx =$$

$$\int_{N_i(\Gamma\cap N)\backslash G}\left\{\sum_{\delta\in\Gamma\cap N_i\backslash\Gamma\cap N}\int_{\Gamma\cap N_i\backslash N_i}\varphi(n\delta x)\sum_{\zeta\in\Gamma\cap N_i} f(g^{-1}\zeta\delta nx)dn\right\}dx =$$

$$\int_{N_i(\Gamma\cap N)\backslash G}\int_{\Gamma\cap N_i\backslash N_i}\varphi(nx)\sum_{\delta\in\Gamma\cap N} f(g^{-1}\delta nx)\mathrm{dndx}.$$

We now use (1) and continue the calculation getting

$$\int_{N_i(\Gamma\cap N)\backslash G}\int_{\Gamma\cap N_i\backslash N_i}\varphi(nx)\sum_{\delta\in\Gamma\cap N}\left\{f(g^{-1}\delta nx)-f(g^{-1}\delta x)\right\}\text{dndx}.$$

Set

$$\Psi(g,x)=\sum_{\delta\in\Gamma\cap N}f(g^{-1}\delta x)$$

and $F(g,n,x)=\Psi(g,nx)-\Psi(g,x)$. We note that $\Psi(g,\delta x)=\Psi(g,x)$ for $\delta\in\Gamma\cap N_i$, $g,x\in G$. Thus if we wish to estimate $F(g,n,x)$ then we may assume that there is a compact subset ω_1 of N such that $n\in\omega_1\cap N_i$ and if $x=n_1a_1m_1k_1$, $n_1\in N$, $a_1\in A$, $m_1\in M$, $k_1\in K$ then $n_1\in\omega_1$. We now assume that $g\in\mathcal{S}=\omega_N A_t^+\omega_M K$ with ω_N compact in N, ω_M compact in M and $A_t^+=\{a\in A|a^\alpha\geq t,\ \alpha \text{ a root of } (P,A)\}$. There is a compact subset ω_2 of N depending only on ω_N and t such that $a^{-1}\omega_N a\subset\omega_2$ for all $a\in A_t^+$. Thus $\mathcal{S}\subset A_t^+\omega_2\omega_M K$. Set $U_1=\omega_2\omega_M K$. Then U_1 is compact. Thus $g=au$ with $a\in A_t^+$ and $u\in U_1$. We also write $x=ax_1(x_1=a^{-1}x)$. With this notation we have

(2)
$$F(g,n,x)=\sum_{\delta\in\Gamma\cap N}\left\{f(u^{-1}a^{-1}\delta nax_1)-f(u^{-1}a^{-1}\delta ax_1)\right\}.$$

Let U be a compact subset of G containing the support of f. Suppose that the term in (2) corresponding to $\delta\in\Gamma\cap N$ is non-zero. Then

$$u^{-1}a^{-1}\delta nax_1\in U \text{ or } u^{-1}a^{-1}\delta ax_1\in U.$$

Hence

$$a^{-1}\delta nax_1\in U_1U \text{ or } a^{-1}\delta ax_1\in U_1U.$$

Then (continuing with the assumption that the term corresponding to δ is not zero) we have

$$a^{-1}\delta naa^{-1}n_1aa^{-1}a_1m_1 \text{ or } a^{-1}\delta aa^{-1}naa^{-1}a_1m_1 \text{ is in } (U_1UK)\cap P.$$

In other words

$$a^{-1}\delta aa^{-1}naa^{-1}n_1aa^{-1}a_1m_1 \text{ or } a^{-1}\delta aa^{-1}n_1aa^{-1}a_1m_1 \text{ is in } (U_1UK)\cap P.$$

If $n'a'm'\in(U_1UK)\cap P$ then $n'\in U_2$ a compact subset of N and $a'\in U_3$ a compact subset of A. Thus

$$a^{-1}\delta aa^{-1}naa^{-1}n_1a \text{ or } a^{-1}\delta aa^{-1}n_1a \text{ is in } U_2.$$

This implies that $a^{-1}\delta a \in V$, a compact subset of N depending only on t, ω_N, ω_1 and U. We also note for future reference that

(3) If $F(g,n,x) \neq 0$ then $x \in \Omega_1 a \Omega_2 \Omega_3 K$ with Ω_1 a compact subset of N, Ω_2 a compact subset of A and Ω_3 a compact subset of M.

Indeed, $a^{-1}a_1 \in U_3$ by the above.

Let $V_1 \subset N$ be an open neighborhood of 1 with compact closure such that if $\gamma \in \Gamma \cap N$ and $\gamma V_1 \cap V_1 \neq \emptyset$ then $\gamma = 1$. Let V_2 be a compact subset of N such that $a^{-1}V_1 a \subset V_2$ for all $a \in A_t^+$. If the term corresponding to δ in the sum (2) is non-zero then $\delta \in aVa^{-1} \subset aVa^{-1}V_1 \subset a^{-1}VV_2 a$. Now

$$vol(aVV_2a^{-1}) \geq vol(aVa^{-1}V_1) \geq$$

$$\sum_{aVa^{-1}\cap\Gamma} vol(V_1) = vol(V_1)|\Gamma \cap aVa^{-1}|.$$

Thus

$$|\Gamma \cap aVa^{-1}|vol(V_1) \leq vol(aVV_2a^{-1}) = a^{2\rho}vol(VV_2).$$

Here $a^{2\rho} = \det(Ad(a)_{|\mathfrak{n}})$ (as usual). Hence

$$|\Gamma \cap aVa^{-1}| \leq Ca^{2\rho}.$$

We therefore see that the number of non-zero terms in (2) is at most $Ca^{2\rho}$ with C depending only on the support of f, Γ and t.

We now estimate each term. The above argument implies that the $x_1's$ that appear in the non-zero terms lie in a compact subset U_4 depending only on the support of f, Γ an t. Let $X_1, ..., X_d$ be a basis of $\mathfrak{n}_i$ with $Ad(a)X_j = a^{\lambda_j}X_j$ for $a \in A$. We write $n = expX(n)$ with $X(n) \in \mathfrak{n}_i$. Then $X(n) = \sum \sigma_j(n)X_j$ and since we are assuming that $n \in \omega_1$, $|\sigma_i(n)| \leq C_1$. Now

$$f(u^{-1}a^{-1}\delta nax_1) - f(u^{-1}a^{-1}\delta ax_1) = -\int_0^1 \frac{d}{dt}f(u^{-1}a^{-1}\delta exp(tX(n))ax_1)dt.$$

So

$$|f(u^{-1}a^{-1}\delta nax_1) - f(u^{-1}a^{-1}\delta ax_1)| \leq \int_0^1 |\frac{d}{dt}f(u^{-1}a^{-1}\delta exp(tX(n))ax_1)|dt$$

$$= \int_0^1 |R(Ad(x_1^{-1})Ad(a^{-1})X(n))f(u^{-1}a^{-1}\delta exp(tX(n))ax_1)|dt \leq$$

$$\sum_j a^{-\lambda_j}|\sigma_j(n)|\int_0^1 |R(Ad(x_1^{-1})X_j)f(u^{-1}a^{-1}\delta exp(tX(n))ax_1)|dt \leq$$

$$C_2\sum_j a^{-\lambda_j}.$$

With C_2 depending only on f, Γ and t. If $a \in A_t^+$ then $a^{-\lambda_j} \leq C_3 a^{-\alpha_i}$ with C_3 depending only on t. This implies that

$$|F(g,n,x)| \leq C_4 a^{2\rho-\alpha_i} \text{ for } g \in \mathcal{S}.$$

We now return to the integral that we are trying to estimate.

$$I = \int_{N_i(\Gamma\cap N)\backslash G}\int_{\Gamma\cap N_i\backslash N_i}\varphi(nx)F(g,n,x)\text{dndx}.$$

We have seen that if $F(g,n,x) \neq 0$ with $n \in \omega_1$ then $x \in \omega_3 aA_s^+\omega_4 K$ with ω_3 compact in N and ω_4 compact in M and s depends only on f and t (see (3) above). We therefore see that

$$|I| \leq C_4 a^{2\rho-\alpha_i}\int_{\omega_3 aA_s^+\omega_4 K}|\varphi(x)|dx.$$

It is standard from the theory of Siegel sets that if $u \in L^1(\Gamma\backslash G)$ then

$$\int_{\omega_3 aA_s^+\omega_4 K}|u(x)|dx \leq Const.\|u\|_1.$$

Thus if we apply the Schwarz inequality we find that

$$|I| \leq Const.a^{2\rho-\alpha_i}\|\varphi\|_2(\int_{\omega_3 aA_s^+\omega_4 K}dx)^{1/2} \leq$$
$$\text{Const.}a^{2\rho-\alpha_i}\|\varphi\|_2(\textstyle\int_{A_s^+}(ab)^{-2\rho}db)^{1/2} =$$
$$\text{Const.}a^{\rho-\alpha_i}\|\varphi\|_2.$$

Let for $g = namk \in \mathcal{S}$, $\eta(g) = \min_i a^{\rho-\alpha_i}$. If we apply the above inequality to all i, we find that

(4) If $g = namk \in \mathcal{S}$ then $|\pi_\Gamma(f)\varphi(g)| \leq Const.\eta(g)\|\varphi\|_2$. With Const. depending only on t and f (and of course Γ).

We note

(5) η is square integrable on $\mathcal{S}$.

Indeed,

$$\int_S \eta(g)^2 dg = \int_{\omega_N \times A_t^+ \times \omega_M \times K} a^{-2\rho} a^{2\rho} \min_i a^{-2\alpha_i} \text{dndadmdk}.$$

Now $\max_i(\alpha_i(\log a)) \geq \sum_i \alpha_i(\log a)/\ell$. Thus

$$\int_S \eta(g)^2 dg \leq Const. \int_{A_t^+} a^{-(2/\ell)\sum \alpha_i} da < \infty.$$

Now there exist $\mathcal{S}_1, ..., \mathcal{S}_n$ Siegel sets (sets of the form of $\mathcal{S}$) for different minimal cuspidal parabolic subgroups such that $G = \cup_j \Gamma \mathcal{S}_j([B2;\text{Theorem } 13.1])$. Let η_j be defined on $\mathcal{S}_j$ in the same way as η was defined for $\mathcal{S}$. Let γ be a continuous function on $\Gamma\backslash G$ such that

$$C^{-1}\gamma(g) \leq \eta_j(g) \leq C\gamma(g),\ g \in \mathcal{S}_j$$

with $C \geq 1$. Then there exists a positive constant E such that $|\pi_\Gamma(f)\varphi(g)| \leq E\gamma(g)\|\varphi\|_2$ for $g \in \Gamma\backslash G$. The proof of the Lemma is now complete.

As we shall see the key to Savin's argument is to carefully measure the dependence on Γ and on f of the constants appearing in the above argument. The above result and its proof imply

Theorem 3.2. *If $f \in C_c^1(G)$ then $\pi_\Gamma(f)$ is of Hilbert-Schmidt class on ${}^oL^2(\Gamma\backslash G)$ and $f \mapsto \|\pi_\Gamma(f)_{|{}^oL^2(\Gamma\backslash G)}\|_{HS}$ is continuous on $C_c^1(G)$.*

This result combined with some Sobolev theory implies that if f is in $C_c^p(G)$ with $p > n/2 + 1 (n = \dim G/K)$ then $\pi_\Gamma(f)$ is trace class on ${}^oL^2(\Gamma\backslash G)$.

4. Some combinatorial lemmas.

In this section we will collect a few combinatorial lemmas that will be used in the proof of Savin's theorem. It is suggested that on first reading this section be skipped. The motivation for the material of this section will appear in the next one. Let G be an affine algebraic group defined over $\mathbf{Q}$ that is $\mathbf{Q}$-simple. Let P be a minimal parabolic $\mathbf{Q}$-subgroup. Let A be a maximal $\mathbf{Q}$-split torus of P and let $\Phi(P, A)$ be the root system of P with respect to A. Then $\Phi(P, A) \cup (-\Phi(P, A))$ is an irreducible root system in the sense of $[Bou]([B - T])$. Let β be the largest root of $\Phi(P, A)$.

Lemma 4.1. *If $\dim A = \ell$ then there exist $\gamma_i, \delta_i \in \Phi(P, A)$, $i = 1, ..., \ell - 1$ such that $\gamma_i + \delta_i = \beta$ and $\{\gamma_1, \delta_1, ..., \gamma_{\ell-1}, \delta_{\ell-1}\}$ consists of $2\ell - 2$ elements.*

Proof. Let $\alpha_1, ..., \alpha_\ell$ be the simple roots of $\Phi(P, A)$ arranged so that $(\beta, \alpha_1) > 0$ and $\alpha_1 + ... + \alpha_j \in \Phi(P, A)$ for $j = 1, ..., \ell$. Set $\gamma_i = \alpha_1 + ... + \alpha_i$ and $\delta_i = \beta - \delta_i$. We note that $\delta_i \in \Phi(P, A)$ since $(\beta, \gamma_i) > 0$. Suppose that $\delta_i = \gamma_j$. If $i = j$ then $\beta = 2\gamma_j$ hence $j = \ell$. If $i > j$ then $\beta = 2\alpha_1 + ... + 2\alpha_j + \alpha_{j+1} + ... + \alpha_i$ so $i = \ell$. If $i < j$ then as above $j = \ell$. Thus we see that if $i, j < \ell$ then $\delta_i \neq \gamma_j$. The lemma now follows.

Let N be the unipotent radical of P and let X_i be a basis of $\mathfrak{n}$ over $\mathbf{Q}$ such that $Ad(a)X_i = a^{\alpha_i}X_i$ for $a \in A$. Let for $t > 0$, $A_t^+ = \{a \in A_\mathbf{R} | a^\alpha > t,\ \alpha \in \Phi(P, A)\}$.

Lemma 4.2. *If $t, c > 0$ then there exist $C(c,t) > 0$ and $p \geq \ell$ such that if $a \in A_t^+$ and $a^\beta > cn(n > 1)$ then*

$$\prod_{i=1}^{r}\left(\frac{1}{n} + a^{-\alpha_i}\right) \leq C(c,t)n^{-p}.$$

Furthermore, if $p < 2$ then $\dim N = 1$.

Proof.

$$\prod_{i=1}^{r}\left(\frac{1}{n} + a^{-\alpha_i}\right) = \sum c_I n^{|I|-r} a^{-\alpha_{i_1}} \cdots a^{-\alpha_{i_{|I|}}}$$

the sum over all subsets $I \subset \{1, ..., r\}(|I|$ the cardinality of I) and c_I a universal coefficient. We will estimate every term in the above expression. If $r - |I| \geq \ell$ then

$$n^{|I|-r} a^{-\alpha_{i_1}} \cdots a^{-\alpha_{i_{|I|}}} \leq t^{-|I|} n^{-\ell}.$$

If $r - |I| < \ell$ then $|I| > r - \ell$. Set $k = r - |I|$. At most k elements of

$$\{\gamma_1, \delta_1, ..., \gamma_{\ell-1}, \delta_{\ell-1}, \beta\}$$

can be missing from $S = \{\alpha_i | i \in I\}$. Hence S contains either $\{\gamma_{i_1}, \delta_{i_1}, ..., \gamma_{i_{\ell-k}}, \delta_{i_{\ell-k}}\}$ or $\{\gamma_{i_1}, \delta_{i_1}, ..., \gamma_{i_{\ell-k-1}}, \delta_{i_{\ell-k-1}}, \beta\}$ for appropriate $i_1 < i_2 < ...$ This implies that

$$n^{|I|-r} a^{-\alpha_{i_1}} \cdots a^{-\alpha_{i_{|I|}}} \leq (cn)^{-\ell+(r-|I|)} n^{|I|-r} t^{-s}$$

with $s = |I| - 2(r - |I|)$ or $|I| - 2(r - |I|) + 1$.*The* first assertion now follows.

Suppose that $\ell = 1$. Then $\Phi(P, A) = \{\alpha, 2\alpha\}$ or $\{\alpha\}$. If $\Phi(P, A) = \{\alpha\}$ then $p = r$ works in the estimate. If $\Phi(P, A) = \{\alpha, 2\alpha\}$ then the dimension of the α root space is at least 2 so p can be chosen to be at least 2. Thus if p must be taken to be 1 then $r = 1$.

Let $G_{\mathbf{R}}$ denote the real points of G and let $G_{\mathbf{Z}}$ be a $\mathbf{Z}$-form of G. Let $G_{\mathbf{Z}}(n) = \Gamma(n)$ be the principal congruence subgroup of $G_{\mathbf{Z}}$ of level n. Let G be an open subgroup of $G_{\mathbf{R}}$ and let Γ be an arithmetic subgroup of G. Fix a maximal compact subgroup of G. Let $\Gamma_1 \supset \Gamma_2 \supset ...$ be a sequence of subgroups of finite index in Γ such that $\cap\Gamma_j = \{1\}$ and for each j there exists n such that $\Gamma_j \supset \Gamma \cap \Gamma(n)$. If Γ is a congruence subgroup then the Γ_i can be chosen to be congruence subgroups. An example of this is $\Gamma_j = \Gamma \cap \Gamma(j)$.

Let P be a minimal parabolic subgroup of G defined over $\mathbf{Q}$. Let $P = P_{\mathbf{R}} \cap G = NAM$, (Langlands decomposition). Let ω be a compact fundamental domain for NM with respect to $\Gamma_1 \cap NM$ and set $\mathcal{S}_t = \{$nmak $\in G | nm \in \omega$, $a \in A$ with $a^\alpha \geq t$ for $\alpha \in \Phi(P, A)$, $k \in K\}$. Let β be (as above) the largest root of $\Phi(P, A)$. The following result is the key combinatorial result of $[S]$ (with a slightly better estimate in (2) than that of $[S]$).

Proposition 4.3.($[S]$). *Let Ω be a compact subset of G. There exist constants C_1 and C_2 depending only on Ω and t such that if $x = nmak \in \mathcal{S}_t$ then*

(1) *If* $a^{\beta} \leq C_1 n$ *then* $|\{\gamma \in \Gamma \cap \Gamma(n) | x^{-1}\gamma y \in \Omega\}| \leq 1$ *for* $y \in G$.

(2) *If* $a^{\beta} > C_1 n$ *then* $|\{\gamma \in \Gamma \cap \Gamma(n) | x^{-1}\gamma y \in \Omega\}| \leq C_2 a^{2\rho} n^{-p}$ *for* $y \in G$. *Here p is as in Lemma 4.2.*

The proof of this result will involve some preparation. Let Δ be the set of simple roots in $\Phi(P, A)$. Let for each subset $I \subset \Delta$, P_I be the corresponding parabolic subgroup (*e.g.* $P_{\emptyset} = G$, $P_{\Delta} = P$). Let $t_I \in \mathbf{R}$, $t_I > 0$ be such that $t_I \geq t_J$ if $I \subset J$. We define $R_{\Delta} = \mathcal{S}_{t_{\Delta}}$. Assuming that R_J has been defined for $J \supset I$, $J \neq I$, we set S equal to the union of the R_J with J strictly containing I we set (following Savin)

$$R_I = \{\text{nmak } \in \mathcal{S}_t | a^{\alpha} \geq t_I,\, \alpha \in I\} - S.$$

Observe that $R_{\emptyset}$ is compact.

We will also need the following lemma from reduction theory (see $[B2; 12.6, 15.3]$.

Lemma 4.4.

(1) *The set* $\{\gamma \in \Gamma | \mathcal{S}_t \cap \gamma \mathcal{S}_t \neq \emptyset\}$ *is finite.*

(2) *Let* $\gamma \in \Gamma$ *and assume that there exists a sequence* $x_i = n_i m_i a_i k_i \in \mathcal{S}_t$ *with* $\lim_{i \to \infty} a_i^{\alpha} = \infty$ *for* $\alpha \in I$ *and* $\gamma x_i \in \mathcal{S}_t$ *then* $\gamma \in P_I$.

We use this to prove

Lemma 4.5. *Let* Ω *be a compact subset of* G. *Then there exists* j *and a compact subset* ω_1 *of* N *depending only on* Ω *and* t *such that if* $x = nmak \in \mathcal{S}_t$, $y \in G$ *and* $x^{-1}y \in \Omega$ *then*

$$\{\gamma \in \Gamma_j | x^{-1}\gamma y \in \Omega\} \subset a\omega_1 a^{-1}.$$

Proof. We first make the following assertion

(*) There exists a sequence $\{t_I\}_{I \subset \Delta}$ such that if $\gamma \in \Gamma$, $x \in R_I$, $y \in G$ and if $x^{-1}y \in \Omega$, $x^{-1}\gamma y \in \Omega$ then $\gamma \in P_I$.

Indeed, suppose that this assertion is false. Then there exists $I \subset \Delta$ and for each i, $x_i = n_i m_i a_i k_i \in \mathcal{S}_t$, $y_i \in G$, $\gamma_i \in \Gamma - (\Gamma \cap P_I)$ such that $\lim_{i \to \infty} a_i^{\alpha} = \infty$, $\alpha \in I$, $x_i^{-1} y_i \in \Omega$ and $x_i^{-1}\gamma_i y_i \in \Omega$. But then

$$k_i x_i^{-1} \gamma_i x_i = k_i x_i^{-1} \gamma_i y_i y_i^{-1} x_i \in K\Omega(\Omega^{-1}).$$

We note that $K\Omega(\Omega^{-1}) \subset NCMK$ with $C \subset A$ a compact subset (depending only on Ω). Hence $\gamma_i x_i \subset x_i$NCMK $\subset N a_i CMK$. There exists $\gamma_i' \in P \cap \Gamma$ such that $\gamma_i'\gamma_i x_i \in \omega a_i CK$ with ω as above. Now $\omega a_i CK \subset \omega A_r^+ K = \mathcal{S}_r$ and for some $r > 0$ depending only on Ω. Let $u = \min\{r, t\}$ then $(\gamma_i'\gamma_i \mathcal{S}_u) \cap \mathcal{S}_u \neq \emptyset$. Thus there is a finite subset S of Γ such that $\gamma_i'\gamma_i \in S$ by Lemma 4.4 (1). Hence there exists $\gamma \in \Gamma - (\Gamma \cap P_I)$ and an infinite number of indices such that $\gamma x_i \in \mathcal{S}_u$. Now Lemma 4.4 (2) implies a contradiction. Our assertion now follows.

We now prove the Lemma at hand. We set $z_x = a^{-1}x$ for $x = nmak$ as above. Then the set $C_t = \{z_x | x \in \mathcal{S}_t\}$ is compact. Let t_I be as in (*). And let $x \in \mathcal{S}_t$, $y \in G$, $\gamma \in \Gamma$ be such that $x^{-1}y$, $x^{-1}\gamma y \in \Omega$. Then if $x \in R_I$ then $\gamma \in P_I$. Write $\gamma = \gamma_I \gamma^I$ with $\gamma_I \in N_I$, $\gamma^I \in M_I A_I$. Write $x = namk$ (as usual). Then

$$a^{-1}\gamma a = z_x x^{-1}\gamma y y^{-1} x z_x^{-1} \subset C_t \Omega(\Omega^{-1})(C_t^{-1}) \cap P_I.$$

The set $C_t\Omega(\Omega^{-1})(C_t^{-1}) \cap P_I$ is compact hence is contained in $\omega_1\omega_2$ with ω_1 compact in N_I and ω_2 compact in $M_I A_I$. Since $x \in R_I$ we can write $a = a_1 a_2$ with $a_1 \in A_I$, $a_2 \in A \cap M_I$ and the set of all a_2 with $x \in R_I$ has compact closure. Thus the set

$$\bigcup_{x \in R_I} a\omega_2 a^{-1}$$

has compact closure, say, D. This implies that $\gamma^I \in D$. Let p_I be the natural projection of P_I onto P_I/N. Then $p_I(\Gamma \cap P_I)$ is arithmetic in P_I/N. Hence $\{p_I(\gamma) | \gamma \in \Gamma \cap P_I\} \cap p_I(D)$ is finite. Hence there exists j such that if γ is as above $\gamma \in \Gamma_j$ then $\gamma^I = 1$ and $\gamma \in a\omega_1 a^{-1}$. This completes the proof of the Lemma.

Lemma 4.6. *There is a bases $X_1, ..., X_r$ and $X'_1, ..., X'_r$ of $\mathfrak{n}_{\mathbf{Q}}$ such that*

(1) *If $i < j$ then $[X_i, X_j] \subset \sum_{k>j} \mathbf{Q}X_k$, and for each i there exists $\alpha_i \in \Phi(P, A)$ such that $Ad(a)X_i = a^{\alpha_i} X_i$.*

(2) $N \cap \Gamma(n) \supset \exp(n\mathbf{Z}X'_1) \cdots \exp(n\mathbf{Z}X'_r)$.

(3) $N \cap \Gamma(n) \subset \exp(n(\sum \mathbf{Z}X_i))$.

Proof. We may assume that $G \subset GL(p, \mathbf{R})$ and $\Gamma(1) = G \cap GL(p, \mathbf{Z})$. So $\mathfrak{n}_{\mathbf{Q}} \subset M_p(\mathbf{Q})$. Fix a basis, $Y_1, ..., Y_r$ of $\mathfrak{n}_{\mathbf{Q}}$ satisfying (1). We note that $N_{\mathbf{Q}} = \exp(\mathbf{Q}Y_1) \cdots \exp(\mathbf{Q}Y_r)$. $\exp(Y_i) = I + Y_i + Y_i^2/2 + ... + Y^{p-1}/(p-1)!$. Choose $p_i \in \mathbf{Z}$, $p_i > 0$ such that $p_i^j Y_i^j / j! \in M_p(\mathbf{Z})$ for $1 \le j \le p-1$. Then clearly $\exp(n\mathbf{Z}p_i Y_i) \subset N \cap \Gamma(n)$. Set $X'_i = p_i Y_i$. Let $\log(I - X) = -\sum_{1 \le j \le p-1} X^i / i$ for X nilpotent in $M_p(\mathbf{Q})$. Then $\log(\Gamma(1) \cap N) \subset (1/(p-1)!)\mathfrak{n} \cap M_p(\mathbf{Z})$. So $\log(\Gamma(n) \cap N) \subset n(1/(p-1)!)n \cap M_p(\mathbf{Z})$. Let $q_i \in \mathbf{Q} - \{0\}$ be such that $(1/(p-1)!)\mathfrak{n} \cap M_p(\mathbf{Z}) \subset \mathbf{Z}q_1 Y_1 + ... + \mathbf{Z}q_r Y_r$. Choose $X_i = q_i Y_i$.

We now prove the first assertion of Proposition 4.3. We apply the Lemma 4.5 to the $\Gamma(n)$. Let n_1 be such that if $x = namk \in \mathcal{S}_t$, $y \in G$, $\gamma \in \Gamma(n_1)$ and $x^{-1}y \in \Omega$, $x^{-1}\gamma y \in \Omega$ then $\gamma \in a\omega_1 a^{-1}$ with ω_1 depending only on t and Ω. Now let $X_1, ..., X_r$ be as in Lemma 4.6 and set $\Lambda = \mathbf{Z}X_1 + ... + \mathbf{Z}X_r$. Let $\lambda > 0$ be such that

$$\log(\omega_1) \subset \{x_1 X_1 + ... + x_r X_r | |x_i| \le \lambda\} = W.$$

Let $0 < c < 1/\lambda$. Then $c\Lambda \cap W = \{0\}$. There exists a constant $c_t \ge 1$ depending only on t such that $a^\alpha \le c_t a^\beta$, $\alpha \in \Phi(P, A)$. Take $C_1 = c_t^{-1}c$. Thus if $a^\beta \le C_1 n$ then $\gamma = 1$. If $a^\beta \le C_1 n$ and $x^{-1}\gamma_i y \in \Omega$ for $i = 1, 2$ then the above implies that $\gamma_1^{-1}\gamma_2 = 1$. So the first assertion follows. As for the second assertion we use the notation above. If $a^\beta \ge C_1 n$ then

$$|n\Lambda \cap Ad(a)W| \leq C_2 \prod_j (a^{\alpha_j}/n + 1).$$

Thus

$$|a\omega_1 a^{-1} \cap \Gamma(n)| \leq a^{2\rho} C_2 \prod_j (1/n + a^{-\alpha_j})$$

so the second assertion now follows from Lemma 4.2.

5. The Hilbert-Schmidt norm inequality in the non-cocompact case.

We retain the notation of the last part of section 4. Fix a basis $X_1, ..., X_d$ of $\mathfrak{g}$. If $u \in C_c^1(G)$ we define

$$\|u\|_{1,2} = \sum_i \|X_i u\|_2.$$

Let Γ, $\Gamma(n)$ be as in the previous section. Fix $u \in C_c^1(G)$ and set

$$T_n = \pi_{\Gamma(n)\cap\Gamma}(u)_{|{}^oL^2(\Gamma\backslash G)}$$

(notation as in section 3). Let p be as in Lemma 4.2. If β is the largest root of $\Phi(P, A)$ and if Δ is the set of simple roots in $\Phi(P, A)$ then $\beta = \sum_{\alpha\in\Delta} n_\alpha \alpha$. Set $\kappa = \sum_{\alpha\in\Delta} n_\alpha$.

Proposition 5.1. *Let Ω be a compact subset of G. Then given $\epsilon > 0$ there exists a constant, $C_{\Omega,\epsilon}$, depending only on Ω and t such that if $u \in C^1(G)$ and supp $u \subset \Omega$ then*

$$\|T_n\|_{HS}^2 \leq vol(\Gamma(n) \cap \Gamma\backslash G)(\|u\|_2^2 + C_{\Omega,\epsilon} n^{-p+2-2/\kappa+\epsilon} \|u\|_{1,2}^2).$$

Note. $n^{-p+2-2/\kappa} = n^{-r}$ with $r < 0$. This is clear if $p \geq 2$. If $p < 2$ then Lemma 4.2 implies that $\Phi(P, A) = \{\beta\}$. Thus $-p + 2 - 2/\kappa = -p$ which in this case is -1.

As usual, the proof of this result will take some preparation. Let $B_n = B_{T_n}$ as in section 1. Lemma 2.1 combined with Proposition 1.1 imply that $B_n \in L^1(\Gamma(n) \cap \Gamma\backslash G)$. Since $\Gamma(n) \cap \Gamma$ is normal in $\Gamma(1) \cap \Gamma$ we see that $\Gamma(1) \cap \Gamma$ acts unitarily on ${}^oL^2(\Gamma(n) \cap \Gamma\backslash G)$ by left translation. Thus, since this action commutes with T_n. We see easily that B_n is left $\Gamma(1) \cap \Gamma$ invariant. As in section 2 we see that

(1) $(\Gamma(1) \cap \Gamma : \Gamma(n) \cap \Gamma) \int_{\Gamma(1)\cap\Gamma\backslash G} B_n(x)dx = \|T_n\|_{HS}^2.$

Let $P^1, ..., P^r$ be a complete set of $\Gamma(1) \cap \Gamma$ conjugacy classes of minimal $\mathbf{Q}$-parabolic subgroups of G. Let for each i, S_t^i be a Siegel set corresponding to $P^i as$ in the previous section. We can choose t so that $\cup_i(\Gamma \cap \Gamma(1))S_t^i = G$. Let $f \in^o L^2(\Gamma(n) \cap \Gamma\backslash G)$. Then

(2)

$$T_n f(x) = \int_G f(xy)u(y)dy = \int_G f(y)u(x^{-1}y)dy =$$

$$= \int_{\Gamma(n)\cap\Gamma\backslash G} f(y) \sum_{\gamma\in\Gamma(n)\cap\Gamma} u(x^{-1}\gamma y)dy.$$

Thus
(3)

$$|T_n f(x)| \le \|f\|_2 \left(\int_{\Gamma(n)\cap\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma(n)\cap\Gamma} u(\mathrm{x}^{-1}\gamma y) \right)^2 dy \right)^{1/2}.$$

Let $C_{1,i}$ be the "C_1" if Proposition 4.3 for $\mathcal{S}_t^i$ and Ω. Set $Y_{i,n} = \{x \in \mathcal{S}_t^i | x = nman, a^\beta \ge C_{1,i}n\}$. If $x \in G - \cup_i(\Gamma(1)\cap\Gamma)Y_{i,n} = X_n$ then the sum in the right hand side of (3) consists of at most one term by Proposition 4.3 (1). We therefore see that

(4) If $x \in X_n$ then $|T_n f(x)| \le \|f\|_2\|u\|_2$.

To estimate $T_n f(x)$ for $x \in Y_{i,n}$ we need a clever observation of Savin. Let Δ_i be the set of simple roots for $\Phi(P^i, A^i)$. Let I be a subset of Δ_i. If $z_1, ..., z_m$ is a basis of $\mathbf{n}^i$ set

$$|u|_1 = \max_j \sup_{x\in N_I} |R(z_j)u(x)|$$

for $u \in C^1(N_I^i)$.

Lemma 5.2. *There exists a constant C_3 such that if $h \in C^1(\Gamma(n)\cap\Gamma\cap N_I^i\backslash N_I^i)$ is such that*

$$\int_{\Gamma(n)\cap\Gamma\cap N_I^i\backslash N_I^i} h(u)du = 0$$

then $|h(u)| \le nC_3|u|_1$.

Proof. Let $Y_i = X_i'$ as in Lemma 4.6. Let $\lambda > 0$ be such that if $\mathcal{C}_i = [-\lambda, \lambda]Y_i$ then

$$\exp(n\mathcal{C}_1)\exp(n\mathcal{C}_2)\cdots\exp(n\mathcal{C}_r)$$

contains a fundamental domain $\mathcal{F}$ for $\Gamma(n)\cap\Gamma\cap N_I^i$. Let $u \in \mathcal{F}$. Then we write $u = u_j v_j$ with $u_j \in \exp(n\mathcal{C}_1)\exp(n\mathcal{C}_2)\cdots\exp(n\mathcal{C}_j)$ and $v_j \in \exp(n\mathcal{C}_{j+1})\cdots\exp(n\mathcal{C}_r)$. Then $h(u) - h(0) = h(u_r) - h(u_0) = h(u_r) - h(u_{r-1}) + h(u_{r-1}) - h(u_{r-2}) + ... + h(u_1) - h(u_0)$. Now $u_j = u_{j-1}\exp s X_j$ with $|s| \le n\lambda$. So the mean value theorem implies that $|h(u_j) - h(u_{j-1})| \le n\lambda |R(X_j)h(u_{j-1}\exp(\theta X_j)|$ for some θ with $0 < \theta < n\lambda$. This implies that there is a constant independent of n, c, such that $|h(u) - h(1)| \le nc|h|_1$. Our hypothesis implies that there exists $\tilde{u}_r, \tilde{u}_i \in \mathcal{F}$ such that Re $h(\tilde{u}_r) = 0$ and Im $h(\tilde{u}_i) = 0$. Thus $h(u) = h(u) - h(1) + \mathit{Re}\ h(1) - \mathit{Re}\ h(\tilde{u}_r) + \mathit{Im}\ h(1) - \mathit{Im}\ h(\tilde{u}_i)$. So $|h(u)| \le 3nc|u|_1$. This completes the proof of the lemma.

We now estimate $T_nf(x)$ for $x \in Y_{i,n}$. Let $x = umak$, $u \in N^i$, $m \in M^i$, $a \in A^i$, $k \in K$. Let $\alpha \in \Delta$ be such that a^α is maximal. Let $Q = P^i_{\{\alpha\}} = N_Q M_Q A_Q K$. We set $\Lambda_n = N_Q \cap \Gamma \cap \Gamma(n)$. We note that $u \mapsto T_nf(ux)$, $u \in N_Q$, satisfies the hypothesis of Lemma 5.2. Thus

$$|T_nf(x)| \le nC_3 \max_j \sup_{u\in N_Q} |\frac{d}{dt}_{t=0} T_nf(\text{uexp}(tz_j)x)| =$$

$$nC_3 \max_j \sup_{u\in N_Q} |R(Ad(x^{-1})z_j)T_nf(ux)|$$

Now $Ad(x)^{-1}X = Ad(k)^{-1}Ad(a)^{-1}Ad(mn)^{-1}X$. Since $mn \in \omega_i$, a compact subset of M^iN^i we see that $Ad(x)^{-1}z_j = \sum_q \varphi_{jq}(x)X_q(X_q$ as above) and $|\varphi_{jq}(x)| \le Ca^{-\alpha}$. Here C depends on ω_i. Hence

$$|T_nf(x)| \le$$

$nC_4a^{-\alpha} \max_q \sup_{z\in N_Q} | \int_{\Gamma(n)\cap\Gamma\backslash G} f(y) \sum_{\gamma\in\Gamma(n)\cap\Gamma} L(X_q)u(x^{-1}z^{-1}\gamma y)dy|$.

Let $z \in N^i$. There exists $\gamma' \in P^i \cap \Gamma \cap \Gamma(1)$ such that $x' = \gamma' zx \in \mathcal{S}^i_t$ and $x' = u'm'ak$. We observe that

$$x^{-1}z^{-1}\gamma y = (x')^{-1}\gamma'\gamma(\gamma')^{-1}(\gamma' y).$$

Since $\Gamma(n) \cap \Gamma$ is normal in $\Gamma(1) \cap \Gamma$, Proposition 4.3 (2) implies that the number of $\gamma \in \Gamma(n) \cap \Gamma$ such that $x^{-1}z^{-1}\gamma y \in \Omega$ is at most $C_{2,i}n^{-p}a^{2\rho}$ (here $C_{2,i}$ is the "C_2" for Ω and $\mathcal{S}^i_t$ as in Proposition 4.3 and p is as in Lemma 4.2). This implies that if $x \in Y_{i,n}$ then

$$|T_nf(x)| \le \|f\|_2 C_5 n^{-p/2+1} a^{\rho-\alpha} \|u\|_{1,2}.$$

From this we conclude that

$$\|T_n\|^2_{HS} = (\Gamma(1) \cap \Gamma : \Gamma(n) \cap \Gamma) \int_{\Gamma(1)\cap\Gamma\backslash G} B_n(x)dx \le$$

$$vol((\Gamma(n) \cap \Gamma)\backslash G)\|u\|^2_2 +$$

$$(\Gamma(1) \cap \Gamma : \Gamma(n) \cap \Gamma)C_6 n^{-p+2} \|u\|^2_{1,2} \sum_i \int_{Y_{n,i}\cap A^i} (\max_{\alpha\in\Delta_i} a^\alpha)^{-2} da.$$

We now note that if $a \in A^i \cap Y_{n,i}$ then $C_1 n \le a^\beta \le (\max_{\alpha\in\Delta_i} a^\alpha)^\kappa$. So, $(\max_{\alpha\in\Delta_i} a^\alpha) \ge (C_1 n)^{1/\kappa}$. Hence for each $\epsilon > 0$

$$\int_{Y_{n,i}\cap A^i} (\max_{\alpha\in\Delta_i} a^\alpha)^{-2} da \le$$

$$(C_1 n)^{-2/\kappa+2\epsilon} \int_{Y_{n,i}\cap A^i} (\max_{\alpha\in\Delta_i} a^\alpha)^{-2\epsilon} da \leq$$

$$(C_1 n)^{-2/\kappa+2\epsilon} \int_{(A^i)^+_t} (\max_{\alpha\in\Delta_i} a^\alpha)^{-2\epsilon} da$$

The latter integral converges for each $\epsilon > 0$ and defines a constant $C_{i,\epsilon}$. We conclude that

$$\|T_n\|_{HS}^2 \leq vol((\Gamma(n)\cap\Gamma)\backslash G))(\|u\|_2^2 + n^{-p+2-2/\kappa+2\epsilon}\sum_i C_{6,i}(\epsilon)\|u\|_{1,2}^2).$$

This completes the proof of Proposition 5.1.

Note. The $n^{2\epsilon}$ can be replaced by a power of $1+\log(n)$ using an argument in [S].

6. Limit multiplicities for the discrete series.

We maintain the notation of the previous section. If $\omega \in \hat{G}$ then we set $N_\Gamma(\omega)$ equal to $\dim Hom_G(H_\omega, {}^\circ L^2(\Gamma\backslash G))$ for $(\pi_\omega, H_\omega) \in \omega$. Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ be as in the previous section. Fix a normalization of invariant measure on G. Let $d(\omega)$ equal 0 if ω is not square integrable and if ω is square integrable then $d(\omega)$ will denote the formal degree of ω (as usual).

Theorem 6.1 [S]. *If $\omega \in \hat{G}$ then*

$$\limsup_{i\to\infty} \frac{N_{\Gamma_i}(\omega)}{vol(\Gamma_i\backslash G)} \leq d(\omega).$$

Proof. Let $\varphi \in C_c^1(G)$ be such that $0 \leq \varphi(x) \leq 1$ for all $x \in G$. Let $v \in H_\omega$ be a unit K-finite vector. Set $u(g) = \varphi(g)\langle v, \pi_\omega(g)v\rangle$. We apply Proposition 5.1 to u. We first note that our assumptions imply that

$$\limsup_{i\to\infty} \frac{N_{\Gamma_i}(\omega)}{vol(\Gamma_i\backslash G)} \leq \limsup_{n\to\infty} \frac{N_{\Gamma(n)\cap\Gamma}(\omega)}{vol(\Gamma(n)\cap\Gamma\backslash G)}.$$

Now as in section 2 we have

$$N_{\Gamma(n)\cap\Gamma}(\omega)\|\pi_\omega(u)\|_{HS}^2 \leq \|T_n\|_{HS}^2.$$

Also following the line of reasoning in the cocompact case we have

$$\|T_n\|_{HS}^2 \geq (\int_G \varphi(g)|\langle\pi_\omega(g)v, v\rangle|^2 dg)^2 \geq (\int_G \varphi(g)^2|\langle\pi_\omega(g)v, v\rangle|^2 dg)^2$$

since $0 \leq \varphi(g) \leq 1$. Thus

$$N_{\Gamma(n)\cap\Gamma}(\omega)\|u\|_2^4 \leq \|T_n\|_{HS}^2.$$

We now use Proposition 5.1 with $\epsilon > 0$ small enough that $p - 2 + 2/\kappa > \epsilon$ to see that

$$\frac{N_{\Gamma(n)\cap\Gamma}(\omega)}{vol(\Gamma(n)\cap\Gamma\backslash G)} \leq \frac{1}{\|u\|_2^2} + C_{\Omega,\epsilon} n^{-p+2-2/\kappa+\epsilon} \frac{\|u\|_{1,2}^2}{\|u\|_2^4} \; .$$

This implies that

$$\limsup_{n\to\infty} \frac{N_{\Gamma(n)\cap\Gamma}(\omega)}{vol(\Gamma(n)\cap\Gamma\backslash G)} \leq \frac{1}{\|u\|_2^2} \; .$$

Let $\varphi_j \in C_c^1(G)$, $0 \leq \varphi_j(g) \leq 1$, $g \in G$. Be such that $\varphi_j(g) \leq \varphi_{j+1}(g)$ for $g \in G$ and such that $\lim_{j\to\infty} \varphi_j(g) = 1$ for all $g \in G$. If we set $u_j(g) = \varphi_j(g)\langle v, \pi_\omega(g)v\rangle$ then we have

$$\limsup_{n\to\infty} \frac{N_{\Gamma(n)\cap\Gamma}(\omega)}{vol(\Gamma(n)\cap\Gamma\backslash G)} \leq \frac{1}{\|u_j\|_2^2} \; .$$

So

$$\limsup_{n\to\infty} \frac{N_{\Gamma(n)\cap\Gamma}(\omega)}{vol(\Gamma(n)\cap\Gamma\backslash G)} \leq \lim_{j\to\infty} \frac{1}{\|u_j\|_2^2} = d(\omega).$$

This completes the proof of the theorem.

To complete the proof of Savin's theorem we recall a result of Rohlfs and Speh.

Theorem 6.2 $[R-S]$. *If $\omega_o \in \hat{G}_d$ then we set $\hat{G}(\omega_o) = \{\omega \in \hat{G} |$ the infinitesimal character of ω equals that of $\omega_o\}$. Then there exist integers $c(\omega)$, $\omega \in \hat{G}(\omega_o)$ such that $c(\omega) = 1$ if $\omega \in \hat{G}(\omega_o) \cap \hat{G}_d$ and*

$$\lim_{i\to\infty} \sum_{\omega\in\hat{G}(\omega_o)} c(\omega) N_{\Gamma_i}(\omega)/vol(\Gamma_i\backslash G) = \sum_{\omega\in\hat{G}(\omega)} d(\omega).$$

This result is proved by a detailed study of the contributions to the Euler characteristic of $\Gamma_j\backslash G/K$ with coefficients in the local system corresponding to the finite dimensional representation of G with infinitesimal character equal to that of ω_o. The steps involved are to show that the contributions from the boundary of the Borel-Serre compactification are negligible with respect to the volume. Then to show that the contribution from the residual spectrum is also negligible. They then give an alternative proof of Theorem 2.12 using Harder's Gauss-Bonnet formula. The result then follows from the calculation of twisted continuous cohomology with respect to a discrete series representation. For many application this formula is just as useful as the following one (which is an immediate consequence of Theorems 6.1 and 6.2.)

Theorem 6.3. *If $\omega_o \in \hat{G}_d$ then*

$$\lim_{i\to\infty} N_{\Gamma_i}(\omega_o)/vol(\Gamma_i\backslash G) = d(\omega_o).$$

Proof. The order of the set $\hat{G}(\omega)$ is finite. In the limit in Theorem 6.2 the non-square integrable representations contribute 0 by Theorem 6.1. Thus

$$\lim_{i\to\infty} \sum_{\omega\in\hat{G}(\omega_o)\cap\hat{G}_d} N_{\Gamma_i}(\omega)/vol(\Gamma_i\backslash G) = \sum_{\omega\in\hat{G}(\omega_o)\cap\hat{G}_d} d(\omega).$$

Theorem 6.1 implies that the limsup of each term on the left of the above equation is at most $d(\omega)$. Hence all of the terms must have limsup equal to $d(\omega)$. Also, at least one term must have liminf at least $d(\omega)$. Hence that term must have limit equal to $d(\omega)$. Subtracting the corresponding terms from both sides of the equation we see that there must be another term with liminf at least $d(\omega)$, etc.

7. Another application of Proposition 5.1.

We retain the notation of the previous section. In this section we give a generalization of a result of $[DeG - W]$ on the distribution of the multiplicities of the spherical principal series to the non-cocompact case. It should be clear to the reader that other results of the type of Proposition 6.2 can be proved by the method below. Let (P_o, A_o) be a minimal parabolic pair over $\mathbf{R}$ for G. Let $\hat{G}_K$ denote the set of irreducible unitary representations of G that contain the trivial one dimensional representation of G. Let for $\omega \in \hat{G}_K$, $\nu(\omega) \in (\mathfrak{a}_o)^*_{\mathbf{C}}$ be (up to the action of the Weyl group of A_o) the Harish-Chandra parameter of ω. That is, if $(\pi_\omega, H_\omega) \in \omega$ and if $v \in H_\omega$ is a unit K-fixed vector then $\langle \pi_\omega(g)v, v\rangle = \varphi_{\nu(\omega)}(g)$ with

$$\int_K a(kg)^{\rho+\nu}dk = \varphi_\nu(g).$$

Here $g = na(g)k$, $n \in N_o$, $a \in A_o$ and $k \in K$. Although $\nu(\omega)$ is not well defined, $\|\nu(\omega)\|$ is well defined.

We recall an inequality from $[DeG - W]$.

(1) If $\nu \in \mathfrak{a}_o^*$ then $\varphi_\nu(g) \geq \varphi_o(g)$.

Let $f \in C^\infty(\mathbf{R})$ be such that $f(x) = 1$ for $0 \leq x \leq 1$, $f(x) = 0$ for $x \geq 3/2$ and $0 \leq f(x) \leq 1$ for $x \in \mathbf{R}$. Set $u_r(x) = f(\sigma(x)/r)$. Then $u_r \in C_c^\infty(G)$ and supp $u_r \subset B_{3r/2}$ for $r > 0$.

Lemma 7.1. *If $X \in \mathfrak{g}$, $r > 0$ then $|Xu_r(g)| \leq (C_X/r)(\chi_{B_{3r/2}} - \chi_{B_r^o})$ with C_X a constant depending only on X (not on r) and B_r^o the interior of B_r. Also $|X\varphi_o(g)| \leq C'_X\varphi_o(g)$.*

Proof. $Xu_r(g) = f'(\sigma(x)/r)X\sigma(x)/r$. Now in the proof of [$W$,Lemma 8.5.4] itwas shown that $|X\sigma(x)| \leq C_{1,X}$. The lemma follows if we take C_X to be $\|f'\|_\infty C_{1,X}$. The last inequality is standard (cf. [W,Lemma 5.2.8]).

Lemma 6.1 implies that $\|u_r\varphi_o\|_{1,2} \leq C_2(1 + 1/r)(\int_{B_{3r/2}} \varphi_o(g)^2dg)$

Let $T_{n,r} = \pi_{\Gamma(n)\cap\Gamma}(u_r)_{|{}^oL^2(\Gamma(n)\cap\Gamma\backslash G)}$.

Set $C_{r,\epsilon} = C_{B_{3r/2},\epsilon}$ as in Proposition 5.1. Then Proposition 5.1 implies that

(2)
$$\|T_{n,r}\|^2 \le vol(\Gamma(n)\cap\Gamma\backslash G)(\|u_r\|^2 + C_{r,\epsilon}n^{-p-2-2/\kappa+\epsilon}\|u_r\|^2_{1,2}).$$

On the other hand if $\omega \in \hat{G}_K$ then

$$\|\pi_\omega(u_r)\|^2_{HS} = |\int_G u_r(g)\varphi_{\nu(\omega)}(g)dg|^2.$$

This combined with (1) implies that if $\nu(\omega) \in \mathfrak{a}_o^*$ then

$$\|\pi_\omega(u_r)\|^2_{HS} \ge |\int_G u_r(g)\varphi_0(g)^2 dg|^2 \ge \|u_r\varphi_o\|^4.$$

If we use this in (2) and the observation after Lemma 6.1, we have (with $\hat{G}_{K,R}$ equal to the set of $\omega \in \hat{G}_K$ such that $\nu(\omega) \in \mathfrak{a}_o^*$)

$$\sum_{\omega\in\hat{G}_{K,R}} N_{\Gamma(n)\cap\Gamma}(\omega) \le$$

$$vol(\Gamma(n)\cap\Gamma\backslash G)(1/\|u_r\varphi_o\|^2 + C(n,r,\epsilon)(\int_{B_{3r/2}}\varphi_o(g)^2dg)/\|u_r\varphi_o\|^2)$$

with $C(n,r,\epsilon) = C_{r,\epsilon}(1+1/r)n^{-p+2-2/\kappa-\epsilon}$.

If we argue as in the last section we have

$$\limsup_{n\to\infty} vol(\Gamma(n)\cap\Gamma\backslash G)^{-1}\sum_{\omega\in\hat{G}_{K,R}} N_{\Gamma(n)\cap\Gamma}(\omega) \le 1/\|u_r\varphi_o\|^2.$$

Taking r to ∞ we have proved the following:

Proposition 7.2.

$$\lim_{n\to\infty} vol(\Gamma(n)\cap\Gamma\backslash G)^{-1}\sum_{\omega\in\hat{G}_{K,R}} N_{\Gamma(n)\cap\Gamma}(\omega) = 0.$$

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