POLYNOMIAL DIFFERENTIAL OPERATORS ASSOCIATED WITH HERMITIAN SYMMETRIC SPACES

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Introduction. In his paper [Sh], Shimura introduced a technique involving Laplace transforms to analyse certain important differential operators related to classical hermitian symmetric spaces of tube type. His technique was used (and expanded on) by several authors ([R-S],[F-K],[K-S]) to prove generalizations of these formulas. The latter papers either explicitly of implicitly use Gindikin's gamma function formulas. In this paper is to show how one can derive their results using basically algebraic methods from a commutation formula in [W]. We also show how our technique can be used to remove the condition of tube type and in addition extend the class of differential operators to all of the basic ones.

The simplest examples of the formulas alluded to above is given as follows. Let $M_{n,m} = M_{n,m}(\mathbf{C})$ denote the space of all $n \times m$ matrices over $\mathbf{C}$, with $n \geq m$. We write $X \in M_{n,m}$ as $X = [x_{ij}]$ and take x_{ij} as coordinates. We write ∂_{ij} for the operator of partial differentiation in the ij coordinate. Let $1 \leq p \leq m$. We enumerate all $p \times p$ minors of an $n \times m$ matrix, X, as Δ_k, $k = 1, ..., d$. We set

$$L_p = \sum_k \Delta_k(x_{ij})\Delta_k(\partial_{ij}).$$

Let $G = GL(n, \mathbf{C}) \times GL(m, \mathbf{C})$ act on $M_{n,m}$ by $(u, v)X = uXv^{-1}$. This induces an action of G on $\mathcal{P}(M_{n,m})$, the polynomials on $M_{n,m}$. As is well known this space decomposes, as a representation of G, into irreducible subspaces each of multiplicity 1 and naturally parametrized by m-tuples of integers $\lambda = (\lambda_1, ..., \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$. Let $\mathcal{P}[\lambda]$ be the corresponding isotypic space for G. If $f \in \mathcal{P}[\lambda]$ then

$$L_p f = \mu_p(\lambda_1 + m - 1, ..., \lambda_{m-1} + 1, \lambda_m) f$$

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with

$$\mu_p(x_1, ..., x_m) = \sum_{j=0}^{p} (-1)^j \tau_j(1, ..., m-p)\sigma_{p-j}(x_1, ..., x_m).$$

Here σ_j is the j−th elementary symmetric function and τ_j is defined by the formal identity

$$\frac{1}{\prod_{i=1}^{q}(1-tx_i)} = \sum_{j=0}^{\infty} t^j \tau_j(x_1, ..., x_q).$$

If $n = m = p$ then this is a substantial part of the classical Capelli identity. (This has been pointed out in [K-S] and, in fact, using a result of [Ho] these authors show how to derive the Capelli identity from this formula.) In the last section of this paper we give the generalization of the above formula to every Hermitian symmetric space. The above formula corresponds to the special case of Cartan domains of type I. Related work in the general context of "pre-homogeneous" spaces has been done by Howe and Umeda. They derive direct generalizations of the Capelli identity. A special case of their results gives expressions for the operators L_p for $n = m$ in terms of determinants in polarization operators completely analogous to the original Capelli identity.

We began our study of this circle of ideas upon receiving the preprint of [K-S]. Since the formulas in that paper were quite similar to formulas that we observed years ago for Shapovalov forms, we felt that there might be an interesting connection. This is basically the content of the second section of this article. Upon finishing this work we received a reprint of [F-K]. This paper contains (in some form) all of the results that we had derived for the various Hermitian forms that play a role in this paper. Although, many of the results in [K-S] on the eigenvalues of the differential operators are easy consequences of the results in [F-K], they do not appear in that article. In addition, there is an earlier article [R-S] that also derives related formulas in greater generality. In fact, the inner product formulas for the standard Hermitian form in [F-K] (and in this article) can be derived from the "b-polynomial" formulas in [R-S].

The new results in this paper are in sections 3 and 4. The last section was inspired by a question of Kostant. The answer to this question is the content of Theorem 4.6.

1. The Shapovalov form. Before we begin our analysis of the Shapavolov form we must recall some notation. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{R}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

be a Cartan decomposition. We assume that $(\mathfrak{g}, \mathfrak{k})$ is of hermitian type. That is, there exists $H \in \mathfrak{z}(\mathfrak{k}_C)$ such that $adH_{|\mathfrak{p}_C}$ has eigenvalues 1 and -1 (note that $iH \in \mathfrak{k}$). Let $\mathfrak{p}^{\pm}$ be the ± 1 eigenspace for $adH_{|\mathfrak{p}_C}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$ and set $\mathfrak{h} = \mathfrak{t}_C$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_C$ we choose a system of positive roots, Φ^+, for the root system, $\Phi = \Phi(\mathfrak{g}_C, \mathfrak{h})$ of $\mathfrak{g}_C$ with respect to $\mathfrak{h}$ such that if $\alpha \in \Phi$ and $\alpha(H) > 0$ then $\alpha \in \Phi^+$. Set

$$\Phi_n^+ = \{\alpha \in \Phi | \alpha(H) = 1\} = \{\alpha \in \Phi | (\mathfrak{g}_C)_\alpha \subset \mathfrak{p}^+\}$$

and

$$\Phi_k^+ = \{\alpha \in \Phi^+ | \alpha(H) = 0\} = \Phi^+ \cap \Phi(\mathfrak{k}_C, \mathfrak{h}).$$

Let $\gamma_1, ..., \gamma_l$ be the standard strongly orthogonal roots (as constructed by Harish-Chandra). These roots are found as follows: γ_1 is the unique simple root of Φ^+ contained in Φ_n^+, let

$$\Omega_1 = \{\alpha \in \Phi_n^+ - \{\gamma_1\} | \alpha \pm \gamma_1 \notin \Phi\}.$$

If $\Omega_1 = \emptyset$ then $l = 1$ otherwise let γ_2 be the smallest element in Ω_1. Set

$$\Omega_2 = \{\alpha \in \Omega_1 - \{\gamma_2\} | \alpha \pm \gamma_2 \notin \Phi\}.$$

If $\Omega_2 = \emptyset$ then $l = 2$. etc.

Then l is the rank of $\mathfrak{g}$ over $\mathbb{R}$. Let B denote the Killing form of $\mathfrak{g}$ normalized so that if $(...,...)$ is the dual form to $B_{|\mathfrak{h}}$ then $(\gamma_1, \gamma_1) = 2$. Then (as is well known) $(\gamma_i, \gamma_i) = 2$ for all i. Let $H_{\gamma_i} \in \mathfrak{h}$ be defined by $B(h, H_{\gamma_i}) = \gamma_i(h)$, for $h \in \mathfrak{h}$. Set $\mathfrak{h}^- = \sum_i \mathbb{C}H_{\gamma_i}$. Then (cf. [He1]) if $\alpha \in \Phi_k^+$ then $\alpha_{|\mathfrak{h}^-}$ has one of the following forms

$$0 \tag{1.1}$$

$$\frac{1}{2}(\gamma_j - \gamma_i), 1 \leq i < j \leq l \tag{1.2}$$

$$-\frac{1}{2}\gamma_i, 1 \leq i \leq l. \tag{1.3}$$

If $\alpha \in \Phi_n^+$ and $\alpha \neq \gamma_i$ for any i then $\alpha_{|\mathfrak{h}^-}$ is of one of the following forms

$$\frac{1}{2}(\gamma_i + \gamma_j), 1 \leq i < j \leq l \tag{1.4}$$

$$\frac{1}{2}\gamma_i, 1 \leq i \leq l. \tag{1.5}$$

If roots of the form (1.3) (hence (1.5)) do not occur then $\mathfrak{g}$ is said to be of "tube type". Let $\mathfrak{g}_C^o$ be the sum of $\mathfrak{h}$ and the root spaces corresponding to $\pm\alpha$ for $\alpha = \gamma_i$ or $\pm\alpha$ for $\alpha_{|\mathfrak{h}^-}$ of the form (1.1),(1.2),(1.4). Set $\mathfrak{g}^o = \mathfrak{g} \cap \mathfrak{g}_C^o$. Then (cf. [W]) $\mathfrak{g}^o$ is a reductive subalgebra of $\mathfrak{g}$. Furthermore, $([\mathfrak{g}^o, \mathfrak{g}^o], [\mathfrak{g}^o, \mathfrak{g}^o] \cap \mathfrak{k})$ is a hermitian symmetric pair of tube type. Let $(\mathfrak{g}_p^o)_C$ be the sum of $\mathfrak{h}$ and the root spaces corresponding to roots of the form $\pm\gamma_i$, $1 \le i \le p$ and $\pm\alpha$ for $\alpha_{|\mathfrak{h}^-}$ of the form (1.1), (1.2), (1.4) with $1 \le i < j \le p$. Set $\mathfrak{g}_p^o = (\mathfrak{g}_p^0)_C \cap \mathfrak{g}$. Then $([\mathfrak{g}_p^o, \mathfrak{g}_p^o], [\mathfrak{g}_p^o, \mathfrak{g}_p^o] \cap \mathfrak{k})$ is a hermitian symmetric pair of tube type and $\gamma_1, ..., \gamma_p$ give a canonical system of strongly orthogonal roots for this pair.

One knows that

(1) If $1 \le i < j \le l$ then the number of $\alpha \in \Phi_k^+$ such that $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}(\gamma_j - \gamma_i)$ is equal to $c = c_{\mathfrak{g}}$ independent of i, j.

We now consider the symmetric algebra, $S(\mathfrak{p}^-)$, of $\mathfrak{p}^-$ as a module for $\mathfrak{k}$ under the extension of $adX_{|\mathfrak{p}^-}$, $X \in \mathfrak{k}$. Schmid has shown (as has Kostant) that as a $\mathfrak{k}$-module, $S(\mathfrak{p}^-)$ is multiplicity free and the highest weights are of the form $-n_1\gamma_1 - ... - n_l\gamma_l$ with $n_1 \ge n_2 \ge ... \ge n_l \ge 0$, $n_j \in \mathbf{Z}$, and each occuring. Let $\mathfrak{n}_k^+$ be the sum of the positive rootspaces for $\mathfrak{k}_C$. The above result easily implies that $S(\mathfrak{p}^-)^{\mathfrak{n}_k^+}$ is a polynomial algebra in generators $u_1, ..., u_l$ with u_i of weight $-\gamma_1 - ... - \gamma_i$. Set $\mathfrak{p}_r^- = \mathfrak{p}^- \cap (\mathfrak{g}_r^o)_C$. A critical observation (cf. [W]) is

(2) $u_j \in S(\mathfrak{p}_j^-)$.

We choose for $\alpha \in \Phi$, $E_\alpha \in (\mathfrak{g}_C)_\alpha$ such that $B(E_\alpha, E_{-\alpha}) = 1$. Set

$$\mathfrak{q}^- = \oplus_{\alpha \in \Phi_n^+ - \{\gamma_1, ..., \gamma_l\}} (\mathfrak{g}_C)_{-\alpha}.$$

In [W] it was shown that the u_j can be chosen such that

(3) $u_j = E_{-\gamma_1} \cdots E_{-\gamma_j} + v_j$ with $v_j \in S(\mathfrak{p}^-)\mathfrak{q}^-$.

With this choice of u_j the key result in [W] is

Lemma 1.1. *If $i > j$ then $[E_{\gamma_i}, u_j] = 0$. There exist for each $\alpha \in \Phi_k^+$, $w_\alpha \in S(\mathfrak{p}_{l-1}^-)$ such that*

$$[E_{\gamma_l}, u_l] = u_{l-1}(H_{\gamma_l} + \frac{l-1}{2}c) + \sum_{\alpha \in \Phi_k^+} w_\alpha E_\alpha.$$

On $S(\mathfrak{p}^-)$ we now define two types of sesquilinear forms. Let $\overline{X}$ denote the complex conjugate of $X \in \mathfrak{g}_C$ with respect to $\mathfrak{g}$. To define the first type we observe that the form $\langle X, Y \rangle = B(X, \overline{Y})$ is positive definite on $\mathfrak{p}^-$. On the space of elements homogeneous of degree j, $S^j(\mathfrak{p}^-)$ we put the inner product that is j! times the j-th symmetric power of $\langle \ldots, \ldots \rangle$. We denote this inner product on $S(\mathfrak{p}^-)$ by $\langle \ldots, \ldots \rangle$. It is invariant under the above action of $\mathfrak{k}$. It also has the following interpretation. Using, B, we can identify $\mathfrak{p}^-$ with $(\mathfrak{p}^+)^*$. So $S(\mathfrak{p}^-)$ is then identified with the holomorphic polynomials on $\mathfrak{p}^+$ and $S(\mathfrak{p}^+)$ is identified with the algebra of constant coefficient differential operators on $\mathfrak{p}^+$. We extend complex conjugation on $\mathfrak{p}^-$ to a complex anti-linear isomorphism of $S(\mathfrak{p}^-)$ onto $S(\mathfrak{p}^+)$. Then

$$(f,g)=\overline{g}f(0) \tag{1.6}$$

under all of these identifications.

We now define the other type of form (we will look at a more general variant of it in later sections). The Poincaré-Birkhoff-Witt theorem implies that

$$U(\mathfrak{g}_C)=\mathfrak{U}(\mathfrak{k}_C)\oplus(\mathfrak{p}^-\mathfrak{U}(\mathfrak{g}_C)+\mathfrak{U}(\mathfrak{g}_C)\mathfrak{p}^+).$$

Let p denote the projection of $U(\mathfrak{g}_C)$ onto $U(\mathfrak{k}_C)$ with respect to this direct sum decomposition. If $z\in\mathbf{C}$ then we define a one dimensional representation, τ_z, of $\mathfrak{k}$ on $\mathbf{C}$ by $\tau_z(aH+X)=az$ for $a\in\mathbf{C}$ and $X\in[\mathfrak{k},\mathfrak{k}]$. We observe that since $\mathfrak{p}^\pm$ is commutative $U(\mathfrak{p}^\pm)=S(\mathfrak{p}^\pm)$. If $X\in\mathfrak{g}_C$ then we write $X^*=-\overline{X}$. We extend this to an conjugate linear anti-automorphism of $U(\mathfrak{g}_C)$. Set

$$\langle f,g\rangle_z=\tau_z(p(g^*f)) \tag{1.7}$$

for $f,g\in S(\mathfrak{p}^-)$. We note that these sesquiliear forms are also $\mathfrak{k}$-invariant. We will also need to use the form

$$\langle x,y\rangle_z=\tau_z(p(y^*x)) \text{ for } x,y\in U(\mathfrak{g}_C).$$

We note that

$$\langle x,yw\rangle_z=\langle y^*x,w\rangle_z, x,y,w\in U(\mathfrak{g}_C), \tag{1.8}$$

$$\langle xu,y\rangle_z=\langle x,y\rangle_z\tau_z(u), x,y\in U(\mathfrak{g}_C), u\in U(\mathfrak{k}_C) \tag{1.9}$$

and

$$\langle xZ,y\rangle_z=0, x,y\in U(\mathfrak{g}_C), Z\in\mathfrak{p}^+. \tag{1.10}$$

Let $S(\mathfrak{p}^-)[n_1,...,n_l]$ be the $\mathfrak{k}$-isotypic component with highest weight

$$n_1\gamma_1-n_2\gamma_2-\ldots-n_l\gamma_l,\ n_1\geq n_2\geq\ldots\geq n_l\geq 0,\ n_i\in\mathbf{Z}.$$

The main result of this section (cf. [F-K]) is

Proposition 1.2. *If* $f,g\in S(\mathfrak{p}^-)[n_1,\ldots,n_l]$ *then*

$$\langle f,g\rangle_z=(-1)^{\sum n_i}\prod_{i=1}^{l}\prod_{j=0}^{n_i-1}(z+\frac{i-1}{2}c-j)(f,g).$$

Proof. As observed in [W] the most direct relationship between the two forms is that if $f,g\in S^j(\mathfrak{p}^-)$ then

$$\lim_{z\to\infty}z^{-j}\langle f,g\rangle_z=(-1)^j\langle f,g\rangle.$$

Set

$$q_{n_1,\dots,n_l}(z)=\prod_{i=1}^{l}\prod_{j=0}^{n_i-1}(z+\frac{i-1}{2}c-j)(f,g).$$

The above limit formula implies that it is enough to show that there exists a constant $a_{n_1,\dots,n_l}$ such that

$$\langle f,g\rangle_z=a_{n_1,\dots,n_l}q_{n_1,\dots,n_l}(z)(f,g)$$

for all $f,g\in S(\mathfrak{p}^-)[n_1,\dots,n_l]$ and $z\in\mathbf{C}$. Write

$$k_1=n_1-n_2,\dots,k_{l-1}=n_{l-1}-n_l,k_l=n_l.$$

The decomposition of $S(\mathfrak{p}^-)$ combined with Schur's lemma imply that it is enought to show that there is a constant $b_{n_1,\dots,n_l}$ such that

$$(u_1^{k_1}\cdots u_l^{k_l},u_1^{k_1}\cdots u_l^{k_l})_z=b_{n_1,\dots,n_l}q_{n_1,\dots,n_l}(z). \qquad (*)$$

We carry this out by first calculating

$$\varphi(z)=\langle u_1^{k_1}\cdots u_l^{k_l},u_1^{k_1}\cdots u_{l-1}^{k_{l-1}+k_l}E_{-\gamma_l}^{k_l}\rangle_z.$$

Since, $\overline{E}_{-\gamma_l}=-aE_{\gamma_l}$ for some $a\in\mathbf{C}$, the expression that we are calculating is

$$a^{k_l}\langle E_{\gamma_l}^{k_l}u_1^{k_1}\cdots u_l^{k_l},u_1^{k_1}\cdots u_{l-1}^{k_{l-1}+k_l}\rangle_z$$

by (1.8). Now we apply Lemma 1.1 and (1.10) to find that if $k_l>0$ then

$$\varphi(z)=a^{k_l}\langle E_{\gamma_l}^{k_l-1}u_1^{k_1}\cdots u_{l-1}^{k_{l-1}}[E_{\gamma_l},u_l^{k_l}],u_1^{k_1}\cdots u_{l-1}^{k_{l-1}+k_l}\rangle_z.$$

Now Lemma 1.1 implies that

$$[E_{\gamma_l},u_l^{k_l}]\equiv k_lu_{l-1}u_l^{k_l-1}(H_{\gamma_l}+\frac{l-1}{2}c-k_l+1)$$

modulo $U(\mathfrak{g}_C)\mathfrak{n}_k^+$. If we apply (1.9) we therefore see that

$$\varphi(z) = a^{k_l} k_l (z + \frac{l-1}{2}c - k_l + 1)\langle E_{\gamma_l}^{k_l-1} u_1^{k_1} \cdots u_{l-1}^{k_{l-1}+1} u_l^{k_l-1}, u_1^{k_1} \cdots u_{l-1}^{k_{l-1}+k_l}\rangle_z.$$

We can thus apply the same argument repeatedly and deduce that

$$\varphi(z) = a^{k_l} k_l! \prod_{j=0}^{k_l-1} (z + \frac{l-1}{2}c - j)\langle u_1^{k_1} \cdots u_{l-1}^{k_{l-1}+k_l}, u_1^{k_1} \cdots u_{l-1}^{k_{l-1}+k_l}\rangle_z.$$

Hence, if P is the $\mathfrak{k}$-invariant projection from $S(\mathfrak{p}^-)$ onto $S(\mathfrak{p}^-)[n_1, ..., n_l]$ then

$$P(u_1^{k_1} \cdots u_{l-1}^{k_{l-1}+k_l} E_{-\gamma_l}^{k_l}) = c_{n_1,...,n_l} u_1^{k_1} \cdots u_l^{k_l}$$

with $c_{n_1,...,n_l} \neq 0$. The desired formula $(*)$ above now follows from the obvious induction using observation (2) above.

2. Results of Kostant-Sahi (Shimura, Rubentahler-Schiffman, Faraut-Koranyi). In this section we will show how Proposition 1.2 can be used to derive some beautiful formulas of the above mentioned authors. We retain the notation of the previous section. We extend τ_z to a representation of $U(\mathfrak{k}_C \oplus \mathfrak{p}^+)$ by setting $\tau_z(\mathfrak{p}^+) = 0$. Let $\mathbf{C}_z$ denote the corresponding 1-dimensional $\mathfrak{k}_C \oplus \mathfrak{p}^+$-module. We set

$$N(z) = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C \oplus \mathfrak{p}^+)} \mathbf{C}_z.$$

Then $\langle ..., ...\rangle_z$ is the sesquilinear version of the Shapovalov form for the generalized Verma module N(z). We assume in this section that $\mathfrak{g} = \mathfrak{g}^o$. Then as a $\mathfrak{k}_C$-module, $\mathbf{C}u_l$ is $\mathbf{C}_{-2}$. Proposition 1.2 implies that the maximal proper submodule of $N(-\frac{l-1}{2}c)$ is just $U(\mathfrak{p}^-)u_l \otimes 1$. This implies that the $\mathfrak{k}$-module homomorphism, T, of $N(z-2)$ into $N(z)$ given by $x \otimes 1 \mapsto xu_l \otimes 1$ is a $\mathfrak{g}$-module homomorphism for $z = -\frac{l-2}{2}c$ and $T(N(-\frac{l-1}{2}c - 2))$ is the maximal proper submodule of $N(-\frac{l-1}{2}c)$. We therefore see that if $x, y \in U(\mathfrak{p}^-)$ then

(1) $\langle x, y\rangle_{-\frac{l-1}{2}c-2} = \frac{\langle xu_l, yu_l\rangle_z}{\langle u_l, u_l\rangle_z}\Big|_{z=-\frac{l-1}{2}c}$.

Here we observe that the rational function on the right hand side of the above equation is defined at the point $z = -\frac{l-1}{2}c$. (1) combined with Proposition 1.2 now easily implies the following result.

Proposition 2.1. *Let $x, y \in S(\mathfrak{p}^-)[n_1, ..., n_l]$ then*

$$\frac{\langle u_l x, u_l y\rangle}{\langle u_l, u_l\rangle} = \frac{\prod_{j=1}^{l}(\frac{l-j}{2}c + n_j + 1)}{\prod_{j=1}^{l}(\frac{l-j}{2}c+1)}(x, y).$$

Proof. Proposition 1.2 implies that the left hand side of (1) is

$$(-1)^{\sum n_j}\prod_{j=1}^{l}\prod_{i=0}^{n_j-1}(-\frac{l-1}{2}c-2-j)\langle x, y\rangle.$$

It also implies that (after the obvious division is done) the right hand side of (1) is

$$(-1)^{\sum n_j}\prod_{j=1}^{l}\prod_{i=1}^{n_j}(-\frac{l-1}{2}c-j)\frac{\langle u_l x, u_l y\rangle}{\langle u_l, u_l\rangle}.$$

The result now follows from the obvious division.

Lemma 2.2. $\langle u_l, u_l\rangle = \prod_{j=1}^{l}(\frac{l-j}{2}c+1)$.

Proof. Set $e_l = \prod_j E_{-\gamma_j}$. Then $u_l = e_l + v$ with $v \in S(\mathfrak{p}^-)$ and $\langle v, e_l\rangle = 0$. Set $w = e_l - v/\langle v, v\rangle$. Then $\langle u_l, w\rangle = 0$. We note that $S(\mathfrak{p}^-)[1,...,1] = \mathbb{C}u_l$. Thus since

$$u_l + \langle v, v\rangle w = (1 + \langle v, v\rangle)e_l = \langle u_l, u_l\rangle e_l,$$

we see that if P is the orthogonal projection of $S(\mathfrak{p}^-)$ onto $S(\mathfrak{p}^-)[1,...,1]$ then $P(e_l) = u_l/\langle u_l, u_l\rangle$.

Let Z(1) be the maximal proper $\mathfrak{g}$-submodule of N(1). Let $L(1) = N(1)/Z(1)$. Then the lowest weight space of L(1) is

$$\mathbb{C}u_l \otimes 1 + Z(1) = \mathbb{C}e_l \otimes 1 + Z(1).$$

This implies that

$$e_l \otimes 1 + Z(1) = \langle u_l, u_l\rangle u_l \otimes 1 + Z(1).$$

Hence

$$\langle e_l, e_l\rangle_1 = \langle u_l, u_l\rangle^{-1}\langle e_l, u_l\rangle_1.$$

Now $\langle e_l, e_l\rangle_1 = 1$ and the calculation in the previous section easily implies that

$$\langle e_l, u_l\rangle_1 = \prod_{j=1}^{l}(\frac{l-j}{2}c+1).$$

The result now follows.

We now combine Proposition 2.1 and Lemma 2.2.

Theorem 2.3. *Let $x, y \in S(\mathfrak{p}^-)[n_1, ..., n_l]$. Then*

$$\langle u_l x, u_l y\rangle = \prod_{j=1}^{l}(\frac{l-j}{2}c + n_j + 1).$$

We now apply this theorem to derive the above mentioned results. We set $Z_m = \overline{u}_l^m u_l^m$ as a differential operator on the polynomials on $\mathfrak{p}^+$. We also set $D_m = u_l^m$

$\overline{u}_l^m$ also as a differential operator. Then Z_m and D_m commute with the action of $\mathfrak{k}$. So Schur's Lemma implies that

$$Z_{m|S(\mathfrak{p}^-)[n_1,...,n_l]} = \varphi_m(n_1,...,n_l)I$$

and

$$D_{m|S(\mathfrak{p}^-)[n_1,...,n_l]} = \psi_m(n_1,...,n_l)I.$$

We note that if $x, y \in S(\mathfrak{p}^-)[n_1,...,n_l]$ then

$$\langle Z_m x, y\rangle = \langle u_l^m x, u_l^m y\rangle = \langle Z_1 u_l^{m-1} x, u_l^{m-1} y\rangle =$$

$$\varphi_1(n_1+m-1,...,n_l+m-1)\langle u_l^{m-1}, u_l^{m-1}\rangle$$

since

$$u_l^r S(\mathfrak{p}^-)[n_1,...,n_l] = S(\mathfrak{p}^-)[n_1+r,...,n_l+r].$$

We therefore see that

(2) $\varphi_m(n_1,...,n_l) = \prod_{j=0}^{m-1}\varphi_1(n_1+j,...,n_l+j)$.

Also Theorem 2.3 implies that

(3) $\varphi_1(n_1,...,n_l) = \prod_{j=1}^{l}(\frac{l-j}{2}c+n_j+1)$.

Thus (2) and (3) give a complete calculation of φ_m.

We to calculate ψ_m we use the observation

(4) $Z_m Z_m = \overline{u}_l^m\, D_m u_l^m$.

Thus if $f \in S(\mathfrak{p}^-)[n_1,...,n_l]$ then

$$Z_m Z_m f = \psi_m(n_1+m,...,n_l+m)\varphi_m(n_1,...,n_l)f.$$

Thus we see that

(5) $\psi_m(n_1+m,...,n_1+m) = \varphi_m(n_1,...,n_l)$.

Since it is a simple matter to see that $\psi_m(n_1,...,n_l) = 0$ for $n_1 \geq n_2 \geq ... \geq n_l \geq 0$ with $n_l < m$. We have also computed ψ_m. This gives a new proof of the main result in [K-S].

We now show how one can use these results to deduce one of the main results on [R-S],[F-K]. We drop the assumption that $\mathfrak{g} = \mathfrak{g}^o$. Set $(x,n) = \prod_{j=0}^{n-1}(x+j)$. As usual, an empty product is 1.

(6) $\langle u_1^{k_1}\cdots u_l^{k_l}, u_1^{k_1}\cdots u_l^{k_l}\rangle = \prod_{j=1}^{l} k_j! \prod_{i=1}^{l-1}\prod_{m=i}^{l-1}(\frac{i}{2}c+k_{m-i+1}+...+k_m, k_{m+1})$.

To prove this we note that (2) in section (1) implies that if $k_l = 0$ then we can compute the desired expression in $\mathfrak{g}_{l-1}^o$. If $l=1$ then the result is clear. Assume for $l-1$. Then the left hand side of (6) is $(n_i = k_i + \cdots + k_l)$

$$\varphi_{k_l}(n_1-k_l,...,n_{l-1}-k_l,0)\langle u_1^{k_1}\cdots u_{l-1}^{k_{l-1}}, u_1^{k_1}\cdots u_{l-1}^{k_{l-1}}\rangle.$$

Now apply (2),(3) and the inductive hypothesis.

3. The general case. In this section we show how many of the results of the previous section extend to the general case. We therefore drop the assumption that $(\mathfrak{g},\mathfrak{k})$ is of tube type. Set

$$\Lambda_n^+ = \{\alpha \in \Phi_n^+ | \alpha_{|\mathfrak{h}^-} = \tfrac{1}{2}\gamma_j \text{ for some } j\}.$$

Put

$$\mathfrak{v}^- = \oplus_{\alpha\in\Lambda_n^+}(\mathfrak{g}_C)_{-\alpha}.$$

Then $\mathfrak{p}^- = \mathfrak{p}_l^- \oplus \mathfrak{v}^-$. Furthermore we have

(1) $B(X,\overline{Y}) = 0$ for $X\in\mathfrak{p}_l^-$, $Y\in\mathfrak{v}^-$.

We also record the following simple consequence of (1).

Lemma 3.1. $\{f\in S(\mathfrak{p}^-)|\langle f, S(\mathfrak{p}_l^-)\rangle = 0\} = S(\mathfrak{p}^-)\mathfrak{v}^-$.

The following lemma is critical to our generalization of the results of the previous section.

Lemma 3.2. *Let $1\le j\le l$ and let $n_1\ge n_2\ge ...\ge n_j\ge 0$. Then*

$$S(\mathfrak{p}^-)[n_1,...,n_j,0,...,0]\cap S(\mathfrak{p}_j^-) = S(\mathfrak{p}_j^-)[n_1,...,n_j].$$

In particular, $S(\mathfrak{p}^-)[m,...,m]\cap S(\mathfrak{p}_l^-) = \mathbf{C}u_l^m$.

Proof. Set $\mathfrak{k}_j = \mathfrak{k}\cap\mathfrak{g}_j^o$. Then $S(\mathfrak{p}^-)[n_1,...,n_l]\cap S(\mathfrak{p}_j^-)$ is $\mathfrak{k}_j$-invariant. Since $S(\mathfrak{p}_j^-)$ decomposes under $\mathfrak{k}_j$ into irreducible pieces with multiplicity 1, this intersection is thus a direct sum of spaces of the form $S(\mathfrak{p}_j^-)[m_1,...,m_j]$. But if $S(\mathfrak{p}_l^-)[m_1,...,m_j]$ occurs in the intersection then

$$u_1^{m_1-m_2}\cdots u_{j-1}^{m_{j-1}-m_j}u_l^{m_j}\in S(\mathfrak{p}^-)[n_1,...,n_j,0,...,0].$$

This is only possible if $m_i = n_i$ for all i.

Let $v_1,...,v_{d(m)}$ be a basis of $S(\mathfrak{p}^-)[m,...,m]$ such that

$$\langle v_i, v_j\rangle = \delta_{i,j}\prod_{p=1}^{m}\prod_{r=1}^{l-1}(\frac{l-r}{2}c+p). \tag{3.1}$$

We set

$$D_m = \sum_j v_j\overline{v}_j. \tag{3.2}$$

Then D_m is a $\mathfrak{k}$-invariant differential operator on $S(\mathfrak{p}^-)$. Our main result of this section is

Theorem 3.3. *Let $n_1 \geq n_2 \geq ... \geq n_l \geq 0$. Then D_m acts on $S(\mathfrak{p}^-)[n_1,...,n_l]$ by $\psi_m(n_1,...,n_l)I$ with*

$$\psi_m(n_1,...,n_l) = \prod_{i=1}^{l}\prod_{j=0}^{m-1}(\frac{l-i}{2}c + n_i - j).$$

Proof. We may assume that v_1 is u_l^m (see (2),(3) in section 2). Since D_m is $\mathfrak{k}$-invariant, Schur's Lemma implies that D_m acts on $S(\mathfrak{p}^-)[n_1,...,n_l]$ by a scalar multiple of the identity which we denote by $\psi_m(n_1,...,n_l)$. Let

$$k_1 = n_1 - n_2, ..., k_{l-1} = n_{l-1} - n_l, k_l = n_l,$$

as usual. Then $u_1^{k_1} \cdots u_l^{k_l} \in S(\mathfrak{p}^-)[n_1,...,n_l] \cap S(\mathfrak{p}_l^-)$ (see (2) in section 1). Thus

$$\langle D_m u_1^{k_1} \cdots u_l^{k_l}, u_1^{k_1} \cdots u_l^{k_l}\rangle = \psi_m(n_1,...,n_l)\langle u_1^{k_1} \cdots u_l^{k_l}, u_1^{k_1} \cdots u_l^{k_l}\rangle.$$

Now, Lemma 3.2 says that

$$S(\mathfrak{p}^-)[m,...,m] \cap S(\mathfrak{p}_l^-) = \mathbf{C}u_l^m.$$

Thus if $j > 1$, $\langle v_j, S(\mathfrak{p}_l^-)\rangle = 0$. Let X_j be a basis of $\mathfrak{v}^-$ then Lemma 3.1 implies that $\overline{v}_j = \sum_i L_{ij}\overline{X}_i$ with L_{ij} a differential operator on $S(\mathfrak{p}^-)$. (1) above implies

$$\overline{X}_i u_1^{k_1} \cdots u_l^{k_l} = 0.$$

Thus

$$\langle D_m u_1^{k_1} \cdots u_l^{k_l}, u_1^{k_1} \cdots u_l^{k_l}\rangle = \langle (u_l^m \overline{u}_l^m) u_1^{k_1} \cdots u_l^{k_l}, u_1^{k_1} \cdots u_l^{k_l}\rangle.$$

We note that all terms in the right hand side of this formula are in $S(\mathfrak{p}_l^-)$. Thus ψ_m is given by the formula in section 2. The result now follows from (2),(3) and (5) in section 2.

There are also generalizations of the operators Z_m of section 2. The simple trick that was used to relate the D_m and the Z_m in the tube type case won't work with out modification. However, let us give a method for analyzing these operators for $m = 1, l = 2$. Thus $Z_1 = \sum_{i=1}^{d(1)} \overline{v}_i v_i$. Let $X_1, ..., X_n$ be a basis of $\mathfrak{p}^-$ such that $\langle X_i, X_j\rangle = \delta_{ij}$. Set $E = \sum X_j \overline{X}_j$ (the usual Euler operator). Then it is easy to see that

$$\sum[\overline{v}_i, v_i] = aI + bE.$$

Thus $Z_1 = aI + bE + D_1$. Clearly $Z_1 1 = a$. Now

$$\langle Z_1 1, 1\rangle = \sum_i \langle v_i, v_i\rangle = d(1)\ (1 + \tfrac{1}{2}c).$$

This computes a. If $X \in \mathfrak{p}^-$ then

$$Z_1 X = aX + bEX = (a+b)X.$$

Hence

$$n(a+b) = \sum_j \langle Z_1 X_j, X_j\rangle = \sum_{j,k} \langle v_k X_j, v_k X_j\rangle =$$

$$\sum_{j,k} \langle X_j v_k, X_j v_k\rangle = \sum_{j,k} \langle X_j X_j v_k, v_k\rangle = nd(1)(1+\frac{1}{2}c) + \sum_k \langle Ev_k, v_k\rangle =$$

$$(n+2)d(1)(1+\frac{1}{2}c).$$

Hence $b = \frac{2d(1)}{n}(1+\frac{1}{2}c)$. This implies

Lemma 3.3. *Assume that $l = 2$. Then Z_1 acts on $S(\mathfrak{p}^-)[\mathfrak{n}_1, \mathfrak{n}_2]$ by*

$$d(1)(1+\frac{c}{2}) + \frac{2d(1)}{n}(1+\frac{c}{2})(n_1+n_2) + \psi_1(n_1,n_2).$$

Here $d(1) = dim S(\mathfrak{p}^-)[1,...,1]$.

Notes. 1. If $d(1) = 1$ (i.e. the tube type case) then the above formula is one for $\varphi_1(n_1, n_2)$ (see (3) in section 2). 2. The two cases of interest for this result are the Hermitian symmetric real form of E_6 and SU(p,2). In these cases we have respectively $d(1) = 10, p(p-1)/2$, $c = 8, 2$ and $n = 16, p(p-1)/2$. This implies that the formulas for the "Z's" are not especially nice in the general case.

4. The operators corresponding to the basic representations. If $v_1, ..., v_d$ is an orthogonal basis of $S(\mathfrak{p}^-)[\mathfrak{n}_1, ..., \mathfrak{n}_l]$ such that (see (6) section 2)

$$\langle v_i, v_i\rangle = \langle u_1^{n_1-n_2}\cdots u_{l-1}^{n_{l-1}-n_l} u_l^{n_l}, u_1^{n_1-n_2}\cdots u_{l-1}^{n_{l-1}-n_l} u_l^{n_l}\rangle$$

then we set

$$D_n = \sum_j v_j \overline{v}_j.$$

Clearly, D_n commutes with the action of $\mathfrak{k}_C$ on $S(\mathfrak{p}^-)$.

Lemma 4.1. (Compare [K-S]) *Fix $n_1 \geq n_2 \geq ... \geq n_l \geq 0$. Then there exists a polynomial on $\mathbb{C}^l$, λ_n, of degree $n_1 + ... + n_l$ such that $\lambda_n(x_{s1}, ..., x_{sl}) = \lambda_n(x_1, ..., x_l)$ for all $s \in S_l$(the symmetric group on l-letters) such that D_n acts on $S(\mathfrak{p}^-)[m_1, ..., m_l]$ by*

$$\lambda_n(m_1 + \tfrac{l-1}{2}c, m_2 + \tfrac{l-2}{2}c, ..., m_l)I.$$

Proof. We set $k_1 = m_1 - m_2, ..., k_l = m_{l-1} - m_l, k_l = m_l$. Let

$$u_m = u_1^{k_1} u_2^{k_2} \cdots u_l^{k_l}.$$

Schur's Lemma implies that $D_{\mathbf{n}}$ acts on $S(\mathfrak{p}^-)[m_1,...,m_l]$ by a scalar multiple, $\mu_{\mathbf{n}}(\mathbf{m})$, of the identity. So in in particular,

$$D_{\mathbf{n}} u_{\mathbf{m}} = \mu_{\mathbf{n}}(\mathbf{m}) u_{\mathbf{m}}.$$

But (2) in section 1 implies that $u_{\mathbf{m}} \in S(\mathfrak{p}_l^-)[\, m_1,...,m_l]$. Lemma 3.2 (applied to the case $j = l$) implies that if we take $v_1,...,v_q$ to be an orthonormal basis of $S(\mathfrak{p}_l^-)[m_1,...,m_l]$ then $\overline{v}_j u_{\mathbf{m}} = 0$ for $j > q$. This implies that we may (and do) assume that $\mathfrak{g} = \mathfrak{g}^o$.

Let $E = E_{\gamma_1}+...+E_{\gamma_l}$. Let K_C be the subgroup of $Aut(\mathfrak{g}_C)$ generated by the e^{adX}, $X \in \mathfrak{k}_C$. Then $K_C \cdot E$ is open in $\mathfrak{p}^+$. We identify $S(\mathfrak{p}^-)$ with, $\mathcal{P}(\mathfrak{p}^+)$, the polynomials on $\mathfrak{p}^+$ (as usual). It now is standard (cf. [S]) that $H = \{k \in K_C | kX = x\}$ is the set of fixed points for an involutive automorphism of K_C, θ_o. Schmid's argument in his proof of the decomposition of $\mathcal{P}(\mathfrak{p}^+)$ into isotypic components proves also that the restriction of $\mathcal{P}$ $(\mathfrak{p}^+)$ to $K_C \cdot E$ is the direct sum of all holomorphic spherical representations of the pair (K_C, θ_o) and the pull-back of $D_{\mathbf{n}}$ corresponds to a K_C-invariant differential operator on K_C/H whose order is the same as that of $D_{\mathbf{n}}$. We note that $\mathfrak{h}^- = \{\mathfrak{h} \in \mathfrak{h} | \theta_o \mathfrak{h} = -\mathfrak{h}\}$ and that the action of the restricted Weyl group of the pair on $\mathfrak{h}^-$ corresponds to the permutations of $\gamma_1,...,\gamma_l$. There exists a constant c_o such that

$$\rho_k(H_{\gamma_j}) = \tfrac{l-j}{2} c + c_o.$$

The result now follows from the Harish-Chandra isomorphism theorem (cf. [He2]).

We will now concentrate on the operators of $D_{\mathbf{n}}$ for $n_1 = 1,...,n_j = 1$ and $n_i = 0$ for $i > j$. We will denote these these operators by L_j, $j = 1,...,l$. Then the operator denoted by D_1 in the previous sections is L_l. we denote by μ_j the symmetric polynomial $\lambda_{\mathbf{n}}$ associated with L_j in Lemma 4.1. The next result gives a formula for μ_j, $j = 1,...,l$. In order to state it we will need some notation. Recall that the basic symmetric functions in l variables are defined by

$$\prod_{j=1}^{l}(t + x_j) = \sum_{j=0}^{l} t^{l-j} \sigma_j(x_1,...,x_l).$$

We also define symmetric functions $\tau_{j,q}$, homogeneous of degree j in q variables by

$$\frac{1}{\prod_{i=1}^{q}(1 - tx_i)} = \sum_{j=0}^{\infty} t^j \tau_{j,q}(x_1,...,x_q).$$

Theorem 4.2. *$\mu_l = \sigma_l$. If $1 \le j < l$ then*

$$\mu_j = \sum_{p=0}^{j} (-\frac{c}{2})^p \tau_{p,l-j}(1,2,...,l-j)\sigma_{j-p}.$$

The proof will take some preparation. We first observe

Lemma 4.3. *Fix $y \in \mathbf{C}$. If f is a polynomial of degree at most j, with $0 \le j \le l$, such that $f(x_{s1},...,x_{sl}) = f(x_1,...,x_l)$ for all $s \in S_l$ and if*

$$f(x_1+y,...,x_j+y,y,...,y) = 0$$

for $(x_1,...,x_j)$ in a Zariski-dense subset of $\mathbf{C}^j$ then $f = 0$.

Proof. If we replace f by g with $g(x_1,...,x_l) = f(x_1+y,...,x_l+y)$ we may assume that $y = 0$. Now, the first fundamental theorem of invariant theory for S_l implies that there exists a polynomial φ on $\mathbf{C}^l$ such that if $u(x_1,...,x_l) = \varphi(x_1, x_2^2, ..., x_l^l)$ then $\deg u = \deg f$ and $f(x) = \varphi(\sigma_1(x),...,\sigma_l(\mathrm{x}))$ for all $x \in \mathbf{C}^l$. Degree considerations imply that φ cannot depend on the last $l-j$ variables. Set $F(x) = (\sigma_1(x),...,\sigma_j(x))$. Then $F(\mathbf{C}^l) = \{F(x,0) | x \in \mathbf{C}^j\}$. Thus if $\varphi(\sigma_1(x,0),...,\sigma_j(x,0)) = 0$ for all $x \in \mathbf{C}^j$ then $\varphi = 0$. Thus if $\varphi(\sigma_1(x,0),...,\sigma_j(x,0)) = 0$ for all x in a Zariski-dense subset of $\mathbf{C}^j$ then $\varphi = 0$. The result now follows.

Lemma 4.4. *Let $1 \le j < l$. Then the restriction of L_j to $S(\mathfrak{p}^-)[m_1,...,m_j,0,..,0]$ is given by $\prod_{i=1}^{j}(m_i + \frac{j-i}{2}c)I$.*

Proof. (2) in secction 1 implies that if $m_1 \ge m_2 \ge ... \ge m_j \ge 0$ then $u_{\mathbf{m}} \in S(\mathfrak{p}_j^-)[m_1,...,m_j]$. Let $\lambda_{\mathbf{m}}$ be the eigenvalue of L_j on $S(\mathfrak{p}^-)[m_1,...,m_j,0,..,0]$. Then $L_j u_{\mathbf{m}} = \lambda_{\mathbf{m}} u_{\mathbf{m}}$. If we apply Lemma 3.2 and the argument in the proof of Theorem 3.3 we see that $L_j u_{\mathbf{m}} = (u_j \overline{u}_j) u_{\mathbf{m}}$. The result now follows from Theorem 3.3.

We now begin the proof of Theorem 4.2. We observe that of $p \le q$ then

$$\sigma_j(x_1,...,x_p,0,...,0) = \sigma_j(x_1,..,x_p)$$

where the σ_j on the left hand side is the elementary symmetric function in q variables. Also, if $p \le q$ then

$$\tau_{j,q}(x_1,...,x_p,0,...,0) = \tau_{j,p}(x_1,...,x_p).$$

We will thus drop the second subscript in τ and just indicate the pertinant variables. We observe that

$$\prod_{i=1}^{l}(t+x_i) = \prod_{i=1}^{j}(t+x_i) \prod_{i=j+1}^{l}(t+x_i).$$

This implies that

(1) $\sigma_q(x_1,...,x_l)=\sum_{p=0}^q \sigma_p(x_{j+1},...,x_l)\sigma_{q-p}(x_1,...,x_j)$.

We also observe that

$$\prod_{i=1}^q(1+tx_i)=\sum_{p=0}^q t^p\sigma_p(x_1,...,x_q).$$

Since

$$\frac{\prod_{i=1}^q(1+tx_i)}{\prod_{i=1}^q(1+tx_i)}=1,$$

we have

(2) $\sum_{p=0}^m(-1)^p\tau_p(x_1,...,x_q)\sigma_{m-p}(x_1,...,x_q)=\delta_{m,0}$ for $0\le m\le q$.

If we combine (2) with (1) we have

(3) $\sigma_j(x_1,...,x_j)=\sum_{p=0}^j(-1)^p\tau_p(x_{j+1},...,x_l)\sigma_{j-p}(x_1,...,x_l)$.

In light of (3), Lemmas 4.1,4.3 and 4.4 imply that

$\mu_j(x_1,...,x_l)=\sum_{p=0}^j(-1)^p\tau_p(-\frac{c}{2},-c,...,-\frac{l-j}{2}c)\sigma_{j-p}(x_1-\frac{l-j}{2}c,...,x_l-\frac{l-j}{2}c)$.

Indeed, we apply (3) to $(m_1+\frac{j-1}{2}c,...,m_j,-\frac{c}{2},...,-\frac{l-j}{2}c)$. We simplify this formula using the homogeneity of τ_p.

(4) $\mu_j(x_1,...,x_l)=\sum_{p=0}^j(\frac{c}{2})^p\tau_p(1,2,...l-j)\sigma_{j-p}(x_1-\frac{l-j}{2}c,...,x_l-\frac{l-j}{2}c)$.

Since $\prod_{i=1}^q(t+s+x_i)=\sum_{p=0}^q t^{q-p}\sigma_p(x_1+s,...,x_q+s)$, we have

$$\sigma_u(x_1+s,...,x_q+s)=\sum_{p=0}^u\binom{q-u+p}{p}s^p\sigma_{u-p}(x_1,...,x_q).$$

We apply this to (4) and have (after a few manipulations of indices)

(5) $\mu_j=\sum_{u=0}^j\left(\sum_{w=0}^u\tau_{u-w}(1,2,...,l-j)(j-l)^w\binom{l-j+u}{w}\right)(\frac{c}{2})^u\sigma_{j-u}$.

We now show how (5) implies the assertion of Theorem 4.2. Observe that

$$\frac{1}{\prod_{i=1}^q(t-x_i)}=\sum_{p=0}^\infty t^{-q-p}\tau_p(x_1,...,x_q).$$

If we replace t with $t+s$ in this expression, apply the binomial theorem and equate coefficients of t then we have

$$\tau_p(x_1-s,...,x_q-s)=\sum_{r=0}^p s^r\binom{-q-p+r}{r}\tau_{p-r}(x_1,...,x_q).$$

If $0\le r\le p$ then

$$\binom{-q-p+r}{r}=(-1)^r\binom{p+q-1}{r}.$$

We use the above two formulas with $s=l-j$, $q=l-j+1$, $x_i=i$, $i=1,...,l-j$, $x_{l-j+1}=0$ to find that

$$\mu_j=\sum_{p=0}^{j}(-\frac{c}{2})^p\tau_p(1,2,...,l-j)\sigma_{j-p}.$$

The proof of Theorem 4.2 is now complete.

We now give a solution to a problem posed by Kostant. We first note a combinatorial lemma.

Lemma 4.5.

$$\sum_{p=0}^{j}(-1)^p\tau_p(x_1,...,x_{l-p})\sigma_{j-p}(x_1,...,x_{l-p-1})=\delta_{0,j}.$$

Proof. By induction on l. If $l=0$ then it is clear. Assume for $l-1$. We now prove the result for l by induction on j. If $j=0$ then the result is clear. If $j=1$ then it just says that $\sigma_1(x_1,...,x_{l-1})=\tau_1(x_1,...,x_{l-1})$. We assume the result for $j-1$ and consider $j>1$. A simple manipulation of generating functions implies

(1) $\tau_p(x_1,...,x_q)=x_1\tau_{p-1}(x_1,...,x_q)+\tau_p(x_2,...,x_q)$.

Consider

$$\sum_{p=1}^{j}(-1)^p\tau_p(x_1,...,x_{l-p})\sigma_{j-p}(x_1,...,x_{l-p-1})=$$

$$-\sum_{p=0}^{j-1}(-1)^p\tau_{p+1}(x_1,...,x_{l-p-1})\sigma_{j-p-1}(x_1,...,x_{l-p-2})=$$

$$-x_1\sum_{p=0}^{j-1}(-1)^p\tau_p(x_1,...,x_{l-p-1})\sigma_{j-p-1}(x_1,...,x_{l-p-2})-$$

$$\sum_{p=0}^{j-1}(-1)^p\tau_{p+1}(x_2,...,x_{l-p-1})\sigma_{j-p-1}(x_1,...,x_{l-p-2}).$$

The first term in the last expression is 0 by the inductive hypothesis. To analyze the second term we observe that

$$\sigma_p(x_1,...,x_q)=\sigma_p(x_2,...,x_q)+x_1\sigma_{p-1}(x_2,...,x_q).$$

This implies that the expression that we are computing is

$$-\sum_{p=0}^{j-1}(-1)^p\tau_{p+1}(x_2,...,x_{l-p-1})\sigma_{j-p-1}(x_2,...,x_{l-p-2})-$$

$$x_1\sum_{p=0}^{j-1}(-1)^p\tau_{p+1}(x_2,...,x_{l-p-1})\sigma_{j-p-2}(x_2,...,x_{l-p-2}) =$$

$$-\sigma_j(x_2,...,x_{l-1}) - x_1\sigma_{j-1}(x_2,...,x_{l-1})$$

by the assumption for $l-1$. The result now follows.

We define a new set of operators

$$C_j = \sum_{u=0}^{j}(\frac{c}{2})^u\sigma_u(1,2,...,l-j-1+u)L_{j-u}.$$

Theorem 4.6. *C_j acts on $S(\mathfrak{p}^-)[\mathfrak{n}_1,...,\mathfrak{n}_l]$ by the scalar* $\sigma_j(n_1+\frac{l-1}{2}c, n_2+\frac{l-2}{2}c,...,n_l)$.

Proof. If $y_1,...,y_l$ are indeterminates then we form the matrix

$$T_l = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\tau_1(y_1,...,y_{l-1}) & 1 & 0 & 0 & \cdots \\ \tau_2(y_1,...,y_{l-2}) & -\tau_1(y_1,...,y_{l-2}) & 1 & 0 & \cdots \\ -\tau_3(y_1,...,y_{l-3}) & \tau_2(y_1,...,y_{l-3}) & -\tau_1(y_1,...,y_{l-3}) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Theorem 4.2 implies that if we set

$$S = \begin{bmatrix} \sigma_o \\ \sigma_1 \\ \sigma_2 \\ \vdots \end{bmatrix}$$

and

$$M = \begin{bmatrix} \mu_o \\ \mu_1 \\ \mu_2 \\ \vdots \end{bmatrix}$$

then $T_l S = M$ if $y_i = \frac{l-i}{2}c$. We assert that the inverse to T_l is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \sigma_1(y_1,\dots,y_{l-1}) & 1 & 0 & 0 & \dots \\ \sigma_2(y_1,\dots,y_{l-1}) & \sigma_1(y_1,\dots,y_{l-2}) & 1 & 0 & \dots \\ \sigma_3(y_1,\dots,y_{l-1}) & \sigma_2(y_1,\dots,y_{l-2}) & \sigma_1(y_1,\dots,y_{l-3}) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

This will imply the lemma. We prove the assertion by induction on l. If $l = 1$ the result is clear. Assume for $l-1$. Then the inverse of T_l is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ z_1 & 1 & 0 & 0 & \dots \\ z_2 & \sigma_1(y_1,\dots,y_{l-2}) & 1 & 0 & \dots \\ z_3 & \sigma_2(y_1,\dots,y_{l-2}) & \sigma_1(y_1,\dots,y_{l-3}) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Lemma 4.5 implies that $z_i = \sigma_i(y_1,\dots,y_{l-1})$. This completes the proof.

References

[F-K] J.Faraut and A.Koranyi,Function Spaces and Reproducing Kernels on Bounded Symmetric Domains, *J. Func. Anal.* **88**(1990),64-89.

[G] S.G.Gindikin, Analysis on homogeneous domains, *Uspekhi Mat. Nauk.***19**,1964, 3-64; English translation: *Russian Math. Surveys* **19**(1964),1-90.

[He1] S.Helgason, "Differential Geometry,Lie Groups and Symmetric Spaces", Academic Press, New York,1978.

[He2] S. Helgason, "Groups and Geometric Analysis," Academic Press, New York, 1984.

[Ho] R.Howe, Remarks on Classical Invariant Theory, *Trans. A. M. S.* **313**(1989), 539-570.

[K-S] B.Kostant and S. Sahi, The Capelli Identity, Tube Domains and the Generalized Laplace Transform, To appear.

[R-S] H.Rubentahler and G.Schiffman, Opérateurs différentiels de Shimura et espaces préhomogènes,*Inv. math.* **90**(1987),409-442.

[S] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitischen Räumen, *Inv. math.* **9** (1969),61-80.

[Sh] G. Shimura, On Differential Operators attached to Certain Reprenstations of Classical Groups, Invent. math. 77(1984),463-488.

[W] N.R.Wallach, The analytic continuation of the discrete series II, *Trans. A.M.S.* **251**, 19-37.