V. THE WEYL ALGEBRA

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§ 1 BERNSTEIN'S INEQUALITY AND SOME CONSEQUENCES

Let K be some fixed field of characteristic zero (usually it will be $\mathbb{C}$ or $\mathbb{R}$). The Weyl algebra $A_n(K) = A_n$ is the algebra of differential operators $\Sigma_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$ with polynomial coefficients acting on $K[x_1,\dots,x_n]$ by formal differentiation. ($x = (x_1,\dots,x_n)$; $\partial = (\partial_1,\dots,\partial_n)$; $\partial_j = \partial/\partial x_j$; $x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$; $\partial^\beta = \partial_1^{\beta_1}\cdot\ldots\cdot\partial_n^{\beta_n}$; $a_{\alpha\beta} \in K$; $a_{\alpha\beta} \neq 0$ for at most finitely many α,β.) One can define $A_n(K)$ as the K algebra generated by $x_1,\dots,x_n,\partial_1,\dots,\partial_n$ subject to the relations $[x_i,x_k] = [\partial_i,\partial_k] = 0$, $[\partial_i,x_k] = \delta_{ik}$, where [,] denotes the commutator and δ_{ik} the Kronecker symbol. There is an automorphism $F : A_n \to A_n$ defined by $x_i \mapsto \partial_i$, $\partial_i \mapsto -x_i$ playing the role of a formal Fourier transformation.

1.1 PROPOSITION. $A_n(K)$ *is a simple algebra, i.e. any two sided ideal in* A_n *is* $\{0\}$ *or* A_n .

Proof. Let $I \subset A_n$ be a two sided ideal and $0 \neq P = \Sigma a_{\alpha\beta} x^\alpha \partial^\beta \in I$. If x_j occurs in P with the highest order $m > 0$, then $0 \neq [\partial_j,P] \in I$ and x_j occurs in $[\partial_j,P]$ with the highest order $m - 1$. (Use $[\partial_j,x_j^k] = kx_j^{k-1}$.) By applying $[\partial_j,\ ]$ m times to P one obtains a nonzero element in I without the variable x_j . Using all the $[\partial_j,\ ]$, $[x_j,\ ]$ gives $1 \in I$. Thus $I = A_n$.

1.2 PROPOSITION. *Let A be a simple ring of infinite length as a left A-module. Then every left A-module of finite length is cyclic i.e. it can be generated by one element.*

(*Proof.* [Bj], pp. 30,31.)

1.3 COROLLARY. *A left* $A_n(K)$*-module of finite length is cyclic.*

1.4 *Filtration of* $A_n(K)$. A filtration of an algebra R is an increasing sequence of linear subspaces

$$\{0\} = S_{-1} \subset S_0 \subset S_1 \subset S_2 \subset \ldots \subset R \quad \text{with}$$

$$\bigcup_j S_j = R \quad \text{and} \quad S_i \cdot S_j \subset S_{i+j} \quad \text{for all } i,j \ .$$

S_o is a subalgebra of R . We shall use two filtrations of A_n :

1) $S_j = T_j = \{\Sigma a_{\alpha\beta} x^\alpha \partial^\beta \mid |\alpha| + |\beta| \leq j\}$, $|\gamma| := \gamma_1 + \ldots + \gamma_n$,

2) $S_j = \Sigma_j = \{\Sigma a_{\alpha\beta} x^\alpha \partial^\beta \mid |\beta| \leq j\}$.

The first filtration is called the Bernstein filtration. Its advantage is that the T_j are finite dimensional. The second filtration can be generalized to algebras of differential operators on varieties; the Σ_j are finite K[x]-modules. ($K[x] = \Sigma_0$.) Observe that $S_i \cdot S_j = S_{i+j}$ for $i,j \geq 0$ in both cases.

Let $F.$ be one of these filtrations. There is an associated graded algebra defined by

$$\mathrm{gr}^F A_n = \bigoplus_{j=0}^{\infty} F_j/F_{j-1} \ , \ [a]\cdot[b] = [ab] \text{ for homogeneous } [a],\ [b] \ .$$

1.5 PROPOSITION. $\mathrm{gr}^F A_n$ *is canonically isomorphic to the polynomial algebra* $K[\bar{x}_1,\ldots,\bar{x}_n,\bar{\partial}_1,\ldots,\bar{\partial}_n]$ *in 2n variables,* $\bar{x}_j, \bar{\partial}_j \in F_1/F_0$.

The proof uses $\deg[x_j,\partial_j] < \deg x_j + \deg \partial_j$ for $\mathcal{T}$ and Σ.

From now on, all modules without explicit declaration as right modules are left modules.

1.6 *Filtrations of* A_n*-modules*. Let M be an A_n-module. A filtration of M is an increasing sequence of K-subspaces

$$\{0\} = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \dots \subset M \text{ , such that}$$

1) $\bigcup_j \Gamma_j = M$,

2) $F_i \cdot \Gamma_j \subset \Gamma_{i+j}$ for all i,j ,

3) the F_0-modules Γ_j (by 2) are finite.

There is an associated graded module

$$gr^{\Gamma} M = \bigoplus_{j=0}^{\infty} \Gamma_j/\Gamma_{j-1} \quad \text{over } gr^{F} A_n$$

with the module structure defined by $[a]\cdot[u] = [au]$ for homogeneous $[a] \in gr^F A_n$, $[u] \in gr^{\Gamma} M$.

1.7 DEFINITIONS . 1) Γ is called a good filtration of M if $gr^{\Gamma} M$ is finitely generated over gr A_n .
2) Let $v \in \Gamma_j - \Gamma_{j-1}$. Then $\sigma(v) = [v] \in \Gamma_j/\Gamma_{j-1}$ is called the symbol of v .

1.8 PROPOSITION. 1) *If* $gr^{\Gamma} M$ *is finitely generated over* gr A_n *, then* M *is finitely generated over* A_n .
2) Γ *is good, if and only if there exists some* j_0 *such that* $F_i \cdot \Gamma_j = \Gamma_{i+j}$ *for all* $i \geq 0$, $j \geq j_0$.
3) *If* Γ *and* Ω *are filtrations of* M *and* Γ *is good, then there exists a* j_1 *, such that* $\Gamma_j \subset \Omega_{j_1+j}$ *for all* j . *If* Γ *and* Ω *are both good, then there exists a* j_2 *, such that* $\Omega_{j-j_2} \subset \Gamma_j \subset \Omega_{j+j_2}$ *for all* j .

Proof. Put $\tilde{\Gamma}_k := \bigoplus_{j\leq k} \Gamma_j/\Gamma_{j-1}$, $\tilde{F}_k := \bigoplus_{j\leq k} F_j/F_{j-1}$.

2) ' $\Rightarrow$ ' : Γ being good means that $gr^{\Gamma}M$ is generated by Γ_{j_0} for some j_0 . Looking at homogeneous components one sees that $\tilde{F}_i \cdot \tilde{\Gamma}_i = \tilde{\Gamma}_{i+j}$ for $i \geq 0$, $j \geq j_0$. Thus $F_i \cdot \Gamma_j + \Gamma_{i+j-1} = \Gamma_{i+j}$ for these i,j . Now use induction on i :

$$F_{i-1} \cdot \Gamma_j = \Gamma_{i+j-1} \Rightarrow F_i \cdot \Gamma_j = F_i \cdot \Gamma_j + \Gamma_{i+j-1} = \Gamma_{i+j} .$$

' $\Leftarrow$ ' : Obviously the condition implies that $gr^{\Gamma}M$ is generated by Γ_{j_0} .

1) Follows from 2 .

3) If $v_1,\dots,v_r$ generate M , then $\Gamma_j := \Sigma_{l=1}^{r} F_j v_l$ defines a filtration of M with the property of 2 .

4) Let j_0 be such that $F_i \cdot \Gamma_j = \Gamma_{i+j}$ for $i \geq 0$, $j \geq j_0$. Since Γ_{j_o} if finite over the (commutative) noetherian ring F_0 and Ω is a filtration of M , $\Gamma_{j_0} \subset \Omega_{j_1}$ for some j_1 . Then

$$\Gamma_j \subset \Gamma_{j_0+j} = F_j \cdot \Gamma_{j_0} \subset F_j \cdot \Omega_{j_1} \subset \Omega_{j_1+j} \qquad \text{for } j \geq 0 .$$

1.9 PROPOSITION . *If* M *is finitely generated over* A_n *, then* M *is noetherian.*

Proof. Let $N \subset M$ be an A_n-submodule and $\Gamma.$ a good filtration of M . (use 1.8.3) Let $\Gamma'_. = \Gamma. \cap N$ be the induced filtration of N . Then $gr^{\Gamma'}N$ is a gr A_n-submodule of the finitely generated module $gr^{\Gamma}M$. Since gr A_n is noetherian as a polynomial ring, $gr^{\Gamma'}N$ is finitely generated, therefore N is finitely generated over A_n by 1.8.1 .

For the remainder of 1. , $F.$ is the Bernstein filtration $T.$.

1.10 *Dimensions and multiplicities* . Let M be a finitely generated $A_n(K)$-module and Γ a good filtration of M . There is a polynomial in t :

$$\chi(M,\Gamma,t) = \frac{m}{d!} t^d + \text{monomials in } t \text{ of smaller degree },$$

$$m,\ d \in \mathbb{N},\ m > 0 ,$$

such that

$$\dim \Gamma_j = \sum_{i=0}^{j} \Gamma_i/\Gamma_{i-1} = \chi(M,\Gamma,j) \text{ for } j \gg 0 .$$

This is the Hilbert polynomial for $\mathrm{gr}^{\Gamma} M$. Since gr A_n is a polynomial ring of 2n variables, $d \leq 2n$.

If Γ, Ω are good filtrations of M , then by 1.8.4

$$\chi(M,\Omega,j-j_2) \leq \chi(M,\Gamma,j) \leq \chi(M,\Omega,j+j_2) .$$

As the leading term of χ is not affected by a shift of j , the numbers $d = d(M)$, $m = m(M)$ depend on M only. They are called the dimension and the multiplicity of M, resp.

Examples. 1) $M = A_n$: $\dim T_j = \binom{2n+j}{2n}$, $d = 2n$, $m = 1$.

2) $M = K[x_1,\ldots,x_n] = A_n/\Sigma_{j=1}^n A_n \partial_j$, $\Gamma.$ = filtration by degrees: $\dim \Gamma_j = \binom{n+j}{n}$, $d = n$, $m = 1$.

3) $M = A_n/\Sigma_{j=1}^n A_n x_j \cong K[\partial_1,\ldots,\partial_n]$: $d = n$, $m = 1$ as in 2. In fact this is the formal Fourier transform of $K[x]$.

1.11 PROPOSITION . *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* A_n*-modules. Let* M *be finitely generated and* Γ *a good filtration of* M . *Let* $\Gamma_.' := \Gamma. \cap M'$ *and* $\Gamma_.'' = \mathrm{Im}\, \Gamma.$ *be the induced filtrations on* M' *and* M", *respectively. Then*

$$0 \to \mathrm{gr}^{\Gamma'} M' \to \mathrm{gr}^{\Gamma} M \to \mathrm{gr}^{\Gamma''} M'' \to 0$$

is an exact sequence of gr A_n*-modules.,* Γ' *and* Γ'' *are good filtrations, and we have*

(1) $\chi(M,\Gamma,t) = \chi(M'.\Gamma',t) + \chi(M'',\Gamma'',t)$;

(2) $d(M) = \max(d(M'), d(M''))$;

(3) *If* $d(M') = d(M'')$, *then* $m(M) = m(M') + m(M'')$.

The proof follows from the fact, that $\mathrm{gr}^{\Gamma}M$ is Noetherian, and therefore Γ' and Γ'' are good filtrations.

1.12 THEOREM (Bernstein). *If* $M \neq \{0\}$ *is a finitely generated* $A_n(K)$*-module, then* $d(M) \geqq n$.

Proof (A. Joseph).Let $\Gamma.$ be a filtration of M with $\Gamma_0 \neq \{0\}$.

LEMMA. *The* K*-linear map*

$$T_i \to \mathrm{Hom}_K(\Gamma_i, \Gamma_{2i}) \ , \ a \mapsto (u \mapsto a\cdot u)$$

is injective .

Proof of the lemma. We proceed by induction on i . The case i = 0 is just the assumption on Γ . Let $0 \neq a \in T_i$. We have to show that $a\cdot\Gamma_i \neq \{0\}$.
Assume, on the contrary, $a\cdot\Gamma_i = \{0\}$, and that the lemma is proved for i-1 . Then a cannot just be constant, so there is some x_j or ∂_j occuring in the expression for a . In the first case, $0 \neq [a,\partial_j] \in T_{i-1}$ and $[a,\partial_j]\cdot\Gamma_{i-1} = \{0\}$, contradicting the i-1 case. In the second case, use x_j instead of ∂_j . This proves the lemma.

Now let Γ be a good filtration of M and χ be the corresponding Hilbert polynomial. From the lemma we obtain the estimate

$$\dim T_i = \frac{1}{(2n)!}i^{2n} + \text{terms of lower order}$$

$$\leqq \chi(i)\cdot\chi(2i) = \dim \mathrm{Hom}_K(\Gamma_i,\Gamma_{2i}) \quad \text{for } i >> 0 \ .$$

Thus, $\deg(\chi) \geqq n$.

1.13 DEFINITION. *A finitely generated* $A_n(K)$*-module* M *is holonomic if* $M \neq 0$ *and* $d(M) = n$ *or if* $M = (0)$.

1.14 COROLLARY. *Submodules and quotients of holonomic modules are holonomic.*

1.15 PROPOSITION. *Let* M *be an* A_n*-module and* $\Gamma.$ *a filtration of* M *with* $\dim_K \Gamma_j \leq \frac{c}{n!}j^n + c_1(j+1)^{n-1}$. *Then* M *is holonomic and* $m(M) \leq c$. *In particular,* M *is finitely generated.*

Proof. Let $N \subset M$ be any finitely generated submodule and $\Omega.$ a good filtration of N . By 1.8.4 there is a j_1 such that $\Omega_j \subset \Gamma_{j+j_1} \cap N \subset \Gamma_{j+j_1}$. From this and Bernstein's theorem we get $\alpha(N) = n$ and then $m(N) \leq c$. If $N_1 \subset N_2 \subset M$, $N_1 \neq N_2$, are both finitely generated, then $m(N_2) - m(N_1) = m(N_2/N_1)$ by 1.11 and 1.12 again. The length of any strictly increasing chain of finitely generated submodules is therefore bounded by c , so M itself is finitely generated.

1.16 *Applications.*

1.16.1 If M is holonomic, then its length is bounded by $m(M)$. (Thus, holonomic modules are cyclic (1.2))
Proof: 1.11 , 1.12 .

1.16.2 THEOREM. *Let* $p \in K[x_1,\dots,x_n]$ *and let* $M = K[x,p(x)^{-1}]$ *have the* A_n*-structure defined by formal differentiation of rational functions. Then* M *is holonomic.*

Proof. Put m = deg p and $\Gamma_j = \{q(x)p(x)^{-j} \mid \deg q \leq (m+1)j\} \subset M$. $\Gamma.$ is a filtration of M . (Proof of $\bigcup \Gamma_j = M$: $q\cdot p^{-j} = q\cdot p^l \cdot p^{-(j+l)} \in \Gamma_{j+l}$ <=> $\deg q + ml \leq (m+1)(j+l)$ <=> $\deg q \leq j+l+mj$, so $q\cdot p^{-j} \in \Gamma_{j+l}$ for $l = \deg q$.) Now use 1.15 .

1.16.3 THEOREM. (Generalization of 1.16.2). *Let* $p \in K[x_1,\dots,x_n]$ *and* M *be a holonomic* A_n*-module. Then* $M[p^{-1}] = K[x,p^{-1}] \otimes_{K[x]} M$ *is holonomic. (The* A_n*-structure is defined by the product rule.)*

Proof. 1.16.3 is proved as 1.16.2 .

1.16.4 *Existence of Bernstein's polynomial. Let* $p \in K[x_1,\dots,x_n]$, $\deg p = m$, and let λ be transcendental over $K[x_1,\dots,x_n]$. Consider the Weyl algebra $A_n(K(\lambda))$ over the function field $K(\lambda)$ and the $A_n(K(\lambda))$-module $N = K(\lambda)[x_1,\dots,x_n,p(x)^{-1}]\cdot p^{\lambda}$ .Here p^{λ} can thought of as a formal symbol, the A_n-action on N is defined by $\partial_j p^{\lambda} = \lambda p^{-1}\cdot p^{\lambda}$ and the product rule (and formal differentiation of rational functions). Let $M = A_n(K(\lambda))\cdot p^{\lambda}$ be the submodule of N generated by p^{λ} . As in 1.16.2, $\Gamma_j := \{q\cdot p^{-j}\cdot p^{\lambda} \mid \deg q \leqq (m+1)j\}$ is a filtration of N with a holonomic estimate (1.15). Therefore, N and M are holonomic. Since M has a finite length, the descending sequence $M_0 \supset M_1 \supset \dots \supset M_j = A_n(K(\lambda))\cdot p^j\cdot p^{\lambda}$, terminates: $M_k = M_{k+1}$ for some k. This means $p^k\cdot p^{\lambda} = D_1(\lambda)\cdot p^{k+1}\cdot p^{\lambda}$, $D_1(\lambda) \in A_n(K(\lambda))$. Since λ is transcendental, we can replace λ by $\lambda-k$ and get $p^{\lambda} = D_1(\lambda-k)\cdot p^{\lambda+1}$. Let $B(\lambda)$ a common multiple of the denominators of the coefficients of $D_1(\lambda-k)$ and put $D(\lambda) = B(\lambda)D_1(\lambda-k)$. Then

$$D(\lambda)\cdot p^{\lambda+1} = B(\lambda)\cdot p^{\lambda} \quad , D(\lambda) \in A_n(K)[\lambda] \ , B(\lambda) \in K[\lambda] \ .$$

$B(\lambda)$ and $D(\lambda)$ are not uniquely defined by this functional equation. The $B(\lambda)$ form an ideal whose generator $b(\lambda)$ with leading coefficient 1 is called the Bernstein polynomial of $p(x)$.

Remark. The functional equation implies $N = M$.

1.16.5 THEOREM. *Let* $p \in K[x_1,\dots,x_n]$ *.Then*

$$M = A_n(K)[\lambda]\cdot p^{\lambda}/A_n(K)[\lambda]\cdot p\cdot p^{\lambda}$$

is a holonomic $A_n(K)$*-module. The miltiplication with* λ *has a minimal polynomial:* $b(\lambda)$.

Proof. The first statement follows from the second one and that is just a reformulation of the results of 1.16.4 .

1.16.6 Let

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n,\mathbb{C}) \mid p_{\alpha\beta}(f) < \infty \text{ for all } \alpha,\beta\}$$

$$p_{\alpha\beta} = (1+|x|^2)^\alpha \cdot |\partial^\beta f| \ , \ \alpha \in \mathbb{N} \ , \ \beta \in \mathbb{N}^n \ ,$$

be the space of tempered functions with its topology defined by the *seminorms* $p_{\alpha\beta}$. *The topological dual space* $S'(\mathbb{R}^n)$ *is the space of* tempered distributions with its weak topology. Let $p(x) \in \mathbb{R}[x_1,\ldots,x_n]$ *have nonnegative values on* $\mathbb{R}^n$ *and let* $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > 0$. *Define*

$$T_\lambda(\phi) = \int_{\mathbb{R}^n} p(x)^\lambda \phi(x)\,dx \ , \ \phi \in S(\mathbb{R}^n) \ .$$

Then $T_\lambda \in S'$ and $\lambda \mapsto T_\lambda$ is holomorphic on $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}$. From $D(\lambda)\cdot p^{\lambda+1} = b(\lambda)\cdot p^\lambda$ we obtain by partial integration

$$\begin{aligned} T_\lambda(\phi) &= b(\lambda)^{-1}\int (D(\lambda)\cdot p^{\lambda+1})\phi\,dx \\ &= b(\lambda)^{-1}\int p^{\lambda+1}\cdot D(\lambda)^*\phi\,dx \\ &= b(\lambda)^{-1} b(\lambda+1)^{-1}\int p^{\lambda+2}\cdot D(\lambda+1)^*\cdot D(\lambda)^*\phi\,dx \\ &= \ldots \end{aligned}$$

This shows that $\lambda \mapsto T_\lambda$ has a meromorphic extension to all of $\mathbb{C}$ with poles in $\{\lambda \mid b(\lambda+j) = 0 \text{ for some } j \in \mathbb{N}\}$.

1.16.7 *Distributional inversion of polynomials*. In 1.16.6, T_λ has a Laurent expansion at $\lambda = -1$:

$$T_\lambda = \sum_{j \geq -N} S_j(\lambda+1)^j \ , \ S_j \in S' \ .$$

Now T_λ is regular at $\lambda = 0$ and $T_0 = 1$: $\int p^\lambda \phi\,dx \to \int \phi\,dx$ for $\lambda \in \mathbb{R}_{>0}$, $\lambda \to 0$. Therefore

$$T_{\lambda+1} = p \cdot T_\lambda = \sum_{j \geq -N} p \cdot S_j (\lambda+1)^j$$

is regular at $\lambda = -1$, thus $p \cdot S_j = 0$ for $j < 0$ and $p \cdot S_0 = 1$.

If $p(x) \in \mathbb{C}[x_1,\dots,x_n]$ is arbitrary, then by defining S_0 with $|p(x)|^2$ instead of $p(x)$ we get $1 = |p|^2 \cdot S_0 = p \cdot (\bar{p} \cdot S_0)$. So $S = \bar{p} \cdot S_0 \in \mathcal{S}'$ satisfies

$$p \cdot S = 1 , \quad S \in \mathcal{S}' .$$

1.16.8 *Fundamental solutions of PDEs with constant coefficients.* Write a differential operator D with constant coefficients in the form $D = p(\frac{1}{i}\partial)$, $p(y) \in \mathbb{C}[y_1,\dots,y_n]$, and let δ be the Dirac distribution, $\delta(\phi) = \phi(0)$. We can use the $S \in \mathcal{S}'(\mathbb{R}^n)$, $p \cdot S = 1$, from 1.16.7 to construct a fundamental solution for $Du = f$, i.e. an $E \in \mathcal{S}'(\mathbb{R}^n)$ with $DE = \delta$:

Define the Fourier transformations on $\mathcal{S}$ and $\mathcal{S}'$ by

$$\phi(x) = \int e^{-i\langle x,y\rangle}\tilde{\phi}(y)dy , \quad \phi \in \mathcal{S} ,$$

$$\tilde{T}(\phi) = T(\tilde{\phi}) \qquad T \in \mathcal{S}' ,$$

and let $E = S$. Using $p \cdot S = 1$ we obtain

$$(p(\tfrac{1}{i}\partial)\tilde{S})(\phi) = \tilde{S}(p(i\partial)\phi) = S(p(i\partial)\phi) = S(p(y)\tilde{\phi})$$

$$= \int \tilde{d}y = \phi(0) = \delta(\phi) .$$

§ 2 SOME HOMOLOGICAL ALGEBRA

2.1 Let M be a finitely generated (left) A_n-module and Γ. a good filtration of M with respect to the Bernstein filtration $\mathcal{T}$. of A_n . The dimension d(M) of 1.10 can be defined in a more geometric way: If $I \subset \mathrm{gr}\, A_n$ is the annihilator ideal of $\mathrm{gr}^{\Gamma} M$ and $J(M) = \sqrt{I}$, then

$$d(M) = \mathrm{Krulldim\ gr}\ A_n/I = \mathrm{Krdim\ gr}\ A_n/J(M) \ .$$

This definition can also be used in the case of the usual filtration Σ. of A_n . If $I' \subset \mathrm{gr}^{\Sigma} A_n$ is the annihilator ideal of $\mathrm{gr}^{\Gamma} M$, Γ. being some Σ-good filtration of M , then $J'(M) = \sqrt{I'}$ (and J(M)) depends only on M (and the choice of the A_n-filtration). (See 2.2 below.)

2.1.1 DEFINITION. 1) The characteristic variety ch(M) of M is the variety defined by J'(M).
2) The (Σ-) dimension of M is defined by

$$\delta(M) = \mathrm{Krdim\ gr}^{\Sigma}\ A_n/J'(M)$$

(If K is algebraically closed, $\delta(M)$ is the dimension of ch(M) .)

One of the aims of this paragraph is to prove the equality $\delta(M) = d(M)$. Then Bernstein's theorem 1.12 says dim ch(M) $\geq$ n in the case of $\bar{K} = K$. This is also an immediate consequence of the much more difficult theorem of the involutivity of ch(M) with respect to some natural symplectic structure on Spec $\mathrm{gr}^{\Sigma} A_n$. The method of proving the equality consists in giving a homological definition for both d(M) and $\delta(M)$ which makes no use of any filtrations on A_n .

2.2. For the remainder of this paragraph we shall work with a general unitary ring R instead of A_n equipped with a filtration ($\{0\} = \Sigma_{-1} \subset \Sigma_0 \subset \Sigma_1 \subset \dots$, $\bigcup_i \Sigma_i = R$, $\Sigma_1 \cdot \Sigma_j \subset \Sigma_{i+j}$ for $i,j \geq 0$; Σ_0 is a subring of R.) We assume for the associated ring $S = \mathrm{gr}^{\Sigma} R$ the following properties:

(i) S is commutative and noetherian.

(ii) S is regular of pure dimension n . (I.e the global homological dimension gl hd$(S_{\mathfrak{m}}) = n$ for all localisation $S_{\mathfrak{m}}$ at maximal ideals $\mathfrak{m} \in S$.)

Because of (i), this is equivalent to

$$\dim_{S/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = n = \text{Krdim } S_{\mathfrak{m}} \quad \text{for all } \mathfrak{m} .$$

This includes the case of A_n , with gl hd(gr A_n) = 2n of course, but also algebras of differential operators on affine varieties. (See the subsequent chapters.) Another case is the ring of differential operators with germs of holomorphic functions as coefficients.

For left or right R-modules, the notions of filtrations and good filtrations are defined as in 1.6 and 1.7 . The propositions 1.8 and 1.9 carry over to this more general situation.

Let M be a finitely generated (left) R-module and Γ. a good filtration of M . Let $I = \text{Ann}_S \text{ gr}^{\Gamma} M \subset S$ and $J(M) = \sqrt{I}$.

LEMMA. J(M) *does not depend on the choice of* Γ .

Proof. I is obviously homogeneous and so is J(M). Given a homogeneous element $\sigma(a) \in S$, $\sigma(a)$ being the symbol of some $a \in R$, $\sigma(a) \in J(M)$ means $a^k \Gamma_j \subset \Gamma_{j-1}$ for some k and all j . Since then $a^{pk}\Gamma_j \subset \Gamma_{j-p}$ for all p , $\sigma(a) \in J(M)$ is equivalent to $\forall r \exists k : a^k \Gamma_j \subset \Gamma_{j-r}$. Because of the compatibility of good filtrations (1.8.4) this is independent of Γ .

2.2.1 DEFINITION. 1) $d(M) := \text{Krdim } (S/J(M))$,

2) $j(M) := \min\{j \,|\, \text{Ext}^j_R(M,R) \neq 0\}$.

2.2.2 THEOREM. $d(M) + j(M) = n$.

The equality $d(M) = \delta(M)$ in 2.1 is an immediate consequence of this theorem, which is proved in the next section.

2.2.3 PROPOSITION. *Let* N *be a finitely generated* S*-module. Let* $d(N) := \mathrm{Krdim}\,(S/\mathrm{Ann}_S N)$ *and* $j(N) := \min\{j \mid \mathrm{Ext}^j_S(N,S) \neq 0\}$ *. Then*

1) $\mathrm{Ext}^j(N,S) = 0$ *for* $j < n - d(N)$,

2) $d(\mathrm{Ext}^j(N,S)) \leqq n - j$ *for all* j ,

3) $d(\mathrm{Ext}^{j(N)}(N,S)) = n - j(N)$.

Proof. 1) and 2) are proposition 5.23, ch.2, in [Bj] .3) is a consequence of 1) and 2) : Assume '<' in 3) . Let $E = \oplus_{j=0}^{n}\mathrm{Ext}^j(N,S)$. The assumption, 1) and 2) imply $d(E) < d(N)$, so there exists an $a \in J(E) - J(N)$. Since N is finitely generated and S noetherian, $\mathrm{Ext}^{\cdot}_S(N,\cdot)$ commutes with localisations. Therefore $N_a \neq 0$ and $\mathrm{Ext}^{\cdot}_{S_a}(N_a,S_a) = 0$. Now let K be any finitely generated S_a-module. From the short exact sequence $0 \to L \to S_a^k \to k \to 0$ and its $\mathrm{Ext}^{\cdot}_{S_a}(N_a,\cdot)$-cohomology sequence we see $\mathrm{Ext}^j_{S_a}(N_a,K) \cong \mathrm{Ext}^{j+1}_{S_a}(N_a,L)$. The assumptions on S imply $\mathrm{Ext}^j_{S_a}(N_a,K) = 0$ for $j > n$. By recursion on j we get $\mathrm{Ext}^j_{S_a}(N_a,K) = 0$ for all j and all finitely generated K. (Observe, that L is finitely generated.) A special case of this is $\mathrm{Hom}_{S_a}(N_a,N_a) = 0$, thus $N_a = 0$. This proves 3).

This proposition gives $d(N) + j(N) = n$ for N a finitely generated S-module. If M is a finitely generated R-module equipped with some good filtration, then $d(M) = d(\mathrm{gr}\,M)$ just by definitions. So theorem 2.2.2 reduces to $j(M) = j(\mathrm{gr}\,M)$.

2.3. *The spectral sequence of a good filtration.*

2.3.1. Let M be a finitely generated left R-module. Given any good filtration $\Gamma.$ of M , there exists $u_1,\dots,u_r \in M$, such that

$$\Gamma_j = \Sigma_{j-m_1}u_1 + \dots + \Sigma_{j-m_r}u_r \quad \text{for all } j \,;$$

$m_k = \deg\sigma(u_k)$. (Namely, choose homogeneous generators v_k of gr M and u_k such that $v_k = \sigma(u_k)$. Then '$\supset$' is obvious and the surjectivity of

$$\Sigma_{j-m_1}u_1 + \dots + \Sigma_{j-m_r}u_r \to \Gamma_j/\Gamma_{j-1}$$

for all j and l is proved by induction on l as in the proof of 1.8.2.) Now let $\varepsilon_1,\dots,\varepsilon_r$ be formal generators of degrees m_k for the free R-module $F_0 = \oplus_{k=1}^{r} R\varepsilon_k$. There is a good filtration of F_0 :

$$\Gamma_j(F_0) = \Sigma_{j-m_1}\varepsilon_1 + \dots + \Sigma_{j-m_r}\varepsilon_r \; .$$

Let K_0 be the kernel of the canonical projection $F_0 \to M$, $\varepsilon_k \mapsto u_k$. Since gr R is noetherian, the induced filtration $\Gamma.(K) = K_0 \cap \Sigma(F_0)$ is good. (gr K_0 = ker(gr $F_0 \to$ gr M) is finitely generated.) Applying this procedure to K_0 instead of M , and so on, one obtains a filtered free resolution of M :

$$\dots \to F_2 \to F_1 \to F_0 \to M \to 0$$

with the properties

1) gr F_j is free over S of the same rank as F_j ,

2) $\dots \to$ gr $F_1 \to$ gr $F_0 \to$ gr M $\to 0$ is exact .

The dual module $N^\vee = \mathrm{Hom}_R(N,R)$ and, more generally, all the $\mathrm{Ext}^j_R(N,R)$ are right R-modules via the right R-module structure of R. If N has the good filtration $\Gamma.(N)$, then $\Gamma_j(N^\vee) := \{\phi \in N^\vee \mid \phi(\Gamma_k(N)) \subset \Sigma_{k+j}$ for all k$\}$ is a good filtration of $N^\vee$. (Proof: Let j_0 be such that $\Sigma_l\Gamma_{j_0}(N) = \Gamma_{j_0+k}(N)$ for all $l \geq 0$. Then $\Gamma_j(N^\vee) = \{\phi \in N^\vee \mid \phi(\Gamma_{j_0}(N)) \subset \Sigma_{j_0+j}\}$. Therefore $\bigcup_j \Gamma_j(N^\vee) = N^\vee$, because $\Gamma_{j_0}(N)$ is finite over Σ_0 , and Σ_0 is noetherian because gr R is. From the injectivity of the canonical gr R-homomorphism gr $N^\vee \to \mathrm{Hom}_{\mathrm{gr}\,R}(\mathrm{gr}\,N, \mathrm{gr}\,R)$ and the fact that gr R is noetherian we see that gr $N^\vee$ is finitely generated.)

We are now going to study the spectral sequence $E^{\cdot}_r$ of the filtered complex $F^\vee_{\cdot}$, the cohomology of which is $\mathrm{Ext}^{\cdot}_R(M,R)$. Since the $F^\vee_j$ are free and finitely generated, the natural maps

$$E^j_0 = \mathrm{gr}\;(F^\vee_j) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{gr}\,R}(\mathrm{gr}\,F_j, \mathrm{gr}\,R)$$

are isomorphisms. So we get

$$E_1^j \cong \mathrm{Ext}^j_{\mathrm{gr}\ R}(\mathrm{gr}\ M, \mathrm{gr}\ R) \ ,$$

$$E_\infty^j \cong \mathrm{gr}\ \mathrm{Ext}^j_R(M,R) \ .$$

For $j < j(\mathrm{gr}\ M)$ we have, by definition of $j(\mathrm{gr}\ M)$, $E_1^j = 0$, and so $E_\infty^j = 0$, because E_∞^j is a subquotient of E_1^j. Thus we get $j(M) \geqq j(\mathrm{gr}\ M)$, and one half of the wanted equality is proved.

In order to prove the other inequality we have to show that $E_\infty^{j(\mathrm{gr}\ M)} \neq 0$.

LEMMA. $E_r^{\cdot}$ *is convergent, i.e.* $E^j_{r(j)} = E^j_\infty$ *for some* $r(j)$.

Proof. a) (A general convergence criterion:) Let $d^j : F_j^\vee \to F_{j+1}^\vee$ be the differentials and let $Z^j = \ker d^j$, $B^j = \mathrm{Im}\ d^{j-1}$. Then $E_r^j = E_\infty^j$, if

$$d^j(F_j^\vee) \cap \Gamma_k F_{j+1}^\vee = d^j(\Gamma_{k+r-1}F_j^\vee) \cap \Gamma_k F_{j+1}^\vee$$

and the analogous equality for $j-1$ instead of j hold.

b) The good filtrations on the $F_j^\vee$ induce two good filtrations on the B^j : The first one (Γ) comes from the inclusion $B^j \subset F_j^\vee$ and the second one (Γ') from the projection $d^{j-1} : F_{j-1}^\vee \to B^j$. From the compatibility of good filtrations (1.8.4) we get an r with $\Gamma_k(B^{j+1}) \subset \Gamma'_{k+r-1}(B^{j+1})$ and $\Gamma_k(B^j) \subset \Gamma'_{k+r-1}(B^j)$ for all k . This is just the condition in a).

Now we use the regularity of $S = \mathrm{gr}\ R$ and proposition 2.2.3. It says:

(i) $E_1^j = 0$ for $j < j(\mathrm{gr}\ M)$,

(ii) $d(E_1^j) \leqq m - j$,

(iii) $d(E^{j(\mathrm{gr}\ M)}) = n - j(\mathrm{gr}\ M)$. Call $j(\mathrm{gr}\ M) = j_0$. Since $E_1^{j_0-1} = 0$, $E_2^{j_0}$ is a submodule of $E_1^{j_0}$, and since $d(E_1^{j_0}) = n - j_0 > d(E_1^{j_0+1})$ we get $d(E_2^{j_0}) = n - j_0$. We also have $E_2^j = 0$ for $j < j_0$ and $d(E_2^j) \leqq n - j$ for all j , because E_2^j is a subquotient of E_1^j . So the properties (i), (ii), (iii) hold for E_2 and in the same way for all E_r^j . Since $E_r^{\cdot}$ converges we have proved the following theorem.

2.3.2 THEOREM. *Let* M *be a finitely generated left* R*-module. Then*

1) $$\mathrm{Ext}_R^j(M,R) = 0 \text{ for } j < j(\mathrm{gr}\ M) ,$$

2) $$d(\mathrm{Ext}_R^j(M,R)) \leq n - j \text{ for all } j ,$$

3) $$d(\mathrm{Ext}_R^{j(\mathrm{gr}\ M)}(M,R)) = n - j(\mathrm{gr}\ M) .$$

An immediate consequence of this is $j(M) = j(\mathrm{gr}\ M)$. Of course, the results of 2.3 hold for right R-modules as well.

2.3.3. THEOREM. *If* M *is holonomic, then* $M^{**} \cong M$.

Proof. From 2.4.3 and the fact that R is noetherian we get a resolution

$$0 \to P_n \to \dots \to P_0 \to M \to 0$$

by finitely generated projective R-modules. Then 2.4.1 yields an exact sequence

$$0 \to P_0^{\vee} \to \dots \to P_n^{\vee} \to M^* \to 0 .$$

2.3.4. *Remark.* From 2.3.3. and Bourbaki Algèbre 10, §8, No3, Cor. of Prop. 4, it follows that the homological dimension of A_n is at most n . From the next theorem, for example, it follows that it is exactly n.

2.4. *Duality for holonomic modules.* Let gr R = S have gl hd(S) = 2n instead of n and assume that Bernstein inequality $d(M) \geq n$ for all finitely generated R-modules $M \neq 0$. The finitely generated (left or right) R-modules of dimension n are called holonomic.

2.4.1 THEOREM - DEFINITION. *A finitely generated left (right) R- module M $\neq 0$ is holonomic if and only if the right (left) R-modules* $\mathrm{Ext}_R^j(M,R) = 0$ *for* $j \neq n$. *In this case,* $M^* = \mathrm{Ext}_R^n(M,R)$ *is called the dual of* M . *It is holonomic.*

Proof. The first statement follows from 2.3.2 (ii), Bernstein's inequality, and 2.2.2 . (Remember that n was replaced by 2n.) The second statement follows from 2.3.2 .

2.4.2 THEOREM. $M \mapsto M^*$ *is an exact functor from the category of holonomic left (right) R-modules to the category of right (left) R-modules.*

Proof. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of holonomic modules then

$$0 \to \mathrm{Ext}_R^n(M_3,R) \to \mathrm{Ext}_R^n(M_2,R) \to \mathrm{Ext}_R^n(M_1,R) \to 0$$

is the only nontrivial part of the long exact $\mathrm{Ext}_R^{\bullet}(\cdot,R)$-cohomology sequence.

2.4.3 THEOREM. *If* M,N *are left (right) R-modules and* M *is finitely generated, then* $\mathrm{Ext}_R^j(M,N) = 0$ *for* $j > n$.

Proof. This is true for $N = R$ by 2.3.2 (ii) and Bernstein's inequality. If N is finitely generated, there is an exact sequence

$$0 \to K \to F \to N \to 0$$

with F free of finite rank and K finitely generated. From the $\mathrm{Ext}_R^{\bullet}(M,\cdot)$-cohomology sequence we get isomorphisms

$$\mathrm{Ext}_R^j(M,N) \cong \mathrm{Ext}_R^{j+1}(M,K) \quad \text{for } j > n .$$

As in the first part of 2.3.1 , $\mathrm{gr}\ \mathrm{Ext}_R^j(M,L)$ $(=E_\infty^j)$ is a subquotient of $\mathrm{Ext}^j_{\mathrm{gr}\ R}(\mathrm{gr}\ M,\mathrm{gr}\ L)$ $(=E_1^j)$ for any finitely generated R-module L and suitable good filtrations of M and L . The latter Ext-groups are all trivial for $j > 2n$ by the assumptions on R. By recursion on j we get the wanted result for a finitely generated N . For the general result one uses that direct limits commute with Ext in the second argument.

If P is a finite projective left (right) module then $P^\vee = \mathrm{Hom}_R(P,R)$ is a finite projective right (left) module. Repeating the first step with M* instead of M leads to

$$0 \to P_n^{\vee\vee} \to \ldots \to P_0^{\vee\vee} \to M^{**} \to 0 .$$

Projective modules are reflexive, thus

$$0 \to P_n \to \ldots \to P_0 \to M^{**} \to 0$$

is exact. This sequence of the P_j is the same as the first one.

2.4.4. *Remark*. From the theorem above it follows, that $M \mapsto M^*$ is an equivalence of categories for holonomic modules.

§ 3 INVERSE AND DIRECT IMAGES OF A_n-MODULES UNDER POLYNOMIAL MAPPINGS.

3.1 *Preservation theorems.* As in the preceding sections modules are usually left modules.

Let M be a holonomic $A_n(K)$-module and H the hyperplane $\{x \in K^n | x_n = 0\}$. Let A_{n-1} be the subalgebra of A_n generated by x_j , ∂_j , $j < n$.

3.1.1 DEFINITION. $\Gamma_{[H]}(M) = \{u \in M | x_n^j u = 0 \text{ for some } j\}$ is the A_n-submodule of M of elements supported by H . (By 1.14 it is also holonomic.)

3.1.2 THEOREM. *Let* M_0 *be the* A_{n-1}*-submodule* $\mathrm{Ker}_M x_n = \{u \in M | x_n u = 0\}$. *Then* $1 \otimes u \mapsto u$ *defines an isomorphism*

$$A_n/A_n x_n \otimes_{A_{n-1}} M_0 \xrightarrow{\sim} \Gamma_{[H]}(M) .$$

($A_n/A_n x_n$ *is considered as a left* A_n*-module and right* A_{n-1}*-module.*)

Proof. Surjectivity: We show $\{u \in M | x_n^k u = 0\} \subset A_n M_0$ by induction on k . If $x_n^k u = 0$, then

$$0 = \partial_n(x_n^k u) = x_n^{k-1}(ku + x_n \partial_n u) .$$

From the induction hypotheses we have $x_n u \in A_n M_0$ and $ku + x_n \partial_n u \in A_n M_0$, therefore

$$(k-1)u = ku + x_n \partial_n u - \partial_n x_n u \in A_n M_0 .$$

The case $k = 1$ is trivial .

Injectivity: The right A_{n-1}-module A_n has the following direct sum decomposition

$$A_n = A_n x_n \oplus A_{n-1} \oplus \partial_n A_{n-1} \oplus \partial_n^2 A_{n-1} \oplus \dots ,$$

so we have

$$A_n/A_n x_n = A_{n-1} \oplus \partial_n A_{n-1} \oplus \partial_n^2 A_{n-1} \oplus \dots \quad .$$

Thus the assertion means that the following sum of A_{n-1}-modules is a direct one and $\partial_n\cdot : \partial_n^k M_0 \to \partial_n^{k+1} M_0$ is injective.

3.1.3 $$\Gamma_{[H]}(M) = M_0 \oplus \partial_n M_0 \oplus \partial_n^2 M_0 \oplus \dots \quad .$$

Proof. Let $u_0 + \partial_n u_1 + \dots + \partial_n^k u_k = 0$, $u_j \in M_0$. Multiplication with x_n gives

$$u_1 + 2\partial_n u_2 + \dots + k\partial_n^{k-1} u_k = 0 \quad .$$

(Use $x_n \partial_n^j = \partial_n^j x_n - j\partial_n^{j-1}$.) Successive multiplication by x_n shows that all $u_j = 0$.

3.1.4 *Remarks*. 1) This proof shows $x_n \cdot \Gamma_{[H]}(M) = \Gamma_{[H]}(M)$.

2) If M' is any finitely generated A_{n-1}-module, then

$$M'' = A_n/A_n x_n \otimes_{A_{n-1}} M'$$

is a finitely generated A_n-module with $d(M'') = d(M') + 1$ and $m(M'') = m(M')$. (In order to see this, write M'' in the form $M'' \cong A_1/A_1 x_n \otimes_K M'$. Here A_1 is the subalgebra of A_n generated by x_n, ∂_n, and the A_n-action on the right hand side is given by the A_1-action on the first and the A_{n-1}-action on the second factor, only. If Γ' is any good filtration of M' and $A_1/A_1 x_n \cong K[\partial_n]$ is filtered or graded by degrees, then $\operatorname{gr} M'' \cong K[\partial_n] \otimes \operatorname{gr} M'$.)

This last remark also proves

3.1.5 THEOREM. *$\operatorname{Ker}_M x_n$ is holonomic.*

3.1.6 THEOREM. *The map $M' \mapsto A_n/A_n x_n \otimes_{A_{n-1}} M'$ defines an equivalence of the category of holonomic A_{n-1}-modules and the category of*

A_n*-modules supported by* H . (I.e. $M'' = \Gamma_{[H]}(M'')$.)

3.1.7 THEOREM. *The* A_{n-1}*-module* M/x_nM *is holonomic.*

Proof. 1) Let $\bar{M} = M/\Gamma_{[H]}(M)$. The canonical A_{n-1}-morphism $M/x_nM \to \bar{M}/x_n\bar{M}$ is an isomorphism : $[u] \mapsto 0$ means $u \in x_nM + \Gamma_{[H]}(M)$; by 3.1.4.1 $u \in x_nM$ follows .
2) We show that $\bar{M}/x_n\bar{M}$ is holonomic. Since the multiplication by x_n is injective on $\bar{M}$, the canonical map $\bar{M} \to \bar{M}[x_n^{-1}]$ is an injective A_n-morphism. Let $N = \bar{m}[x_n^{-1}]/\bar{M}$ .Then $u \mapsto x_n^{-1}u$ defines an A_{n-1}-isomorphism $\bar{M}/x_n\bar{M} \overset{\sim}{\to} Ker_N x_n$. Now the assertion follows from part 1 , 1.14 and 3.1.5 .

3.1.8 THEOREM. $Ker_M \partial_n$ *and* M/∂_nM *are holonomic.*

Proof. This follows from 3.1.5 and 3.1.7 by Fourier-transformation: For any A_n-module N define F(N) to be the A_n-module which is identical to N as a K-vector space but with a different A_n-structure. Namely, let $F : A_n \to A_n$ be the automorphism defined by $x_j \mapsto \partial_j$, $\partial_j \mapsto -x_j$, and define $D\cdot u_{F(N)} := F^{-1}(D)\cdot u_N$. The index of u indicates an element of which module u is considered as. The Fourier-transformation leaves Bernstein's filtration invariant, therefore it preserves holonomicity of A_n-modules.

3.2 *Inverse images.* Let $P = (p_1,\dots,p_n) : K^m \to K^n$ be a polynomial mapping. Then $K[x_1,\dots,x_m]$ is a $K[y_1,\dots y_n]$-module via P . Let M be an A_n-module.

3.2.1 DEFINITION. The inverse image of M under P is the A_m-module

$$P^*M = K[x] \otimes_{K[y]} M$$

with the structure defined by

$$x_j\cdot(q(x)\otimes u) = x_jq(x)\otimes u$$

$$\partial_{x_j}\cdot(q(x)\otimes u) = \frac{\partial q}{\partial x_j}\otimes u + \sum_{i=1}^{n} q\,\frac{\partial p_i}{\partial x_j}\otimes \partial_{y_i} u\ .$$

3.2.2 PROPOSITION. *If* $Q : K^l \to K^m$ *is a second polynomial mapping, then* $(P\circ Q)^*M = Q^*P^*M$.

Proof. This is just a combination of the corresponding statement for modules over polynomial rings and the chain rule.

In order to study P^* , we write P as a composition of an embedding and a projection :

$$P = \pi o i$$

$$i : K^m \to K^{m+n}\ ,\ i(x) = (x,P(x))$$

$$\pi : K^{m+n} \to K^n\ ,\ \pi(x,y) = y$$

$\underline{\pi^*}$: $\pi^*M = K[x,y]\otimes_{K[y]}M \cong K[x]\otimes_K M$. On the right hand side x_j , ∂_{x_j} act on $K[x]$ only and y_j , ∂_{y_j} act on M only. If M is finitely generated with the good filtration $\Gamma.$ and $\Delta.$ is the filtration of $K[x]$ by degrees, then $\Omega_l = \Sigma_{j+k=l}(\Delta_j\otimes\Gamma_k)$ is a good filtration of π^*M . From $gr^\Omega\ \pi^*M = gr^\Delta\ K[x]\otimes_K gr^\Gamma\ M$ one deduces $d(\pi^*M) = m + d(M)$. (Compare 3.1.4.2 for $m = 1$.) This shows that π^*M is holonomic if M is.

$\underline{i^*}$: Let N be an A_{m+n}-module. Then

$$i^*N = K[x]\otimes_{K[x,y]}N \cong K[x,y]/(y_1-p_1,\dots,y_n-p_n)\otimes_{K[x,y]}N,$$

where the A_m-structure on the right hand side is given by

$$x_j\cdot(q(x,y)\otimes u) = x_jq\otimes u\ ,$$

$$\partial_{x_j}\cdot(q(x,y)\otimes u) = \frac{\partial q}{\partial x_j}\otimes u + q\otimes\partial_{x_j}u + \sum_{i=1}^{n}\frac{\partial p_1}{\partial x_j}\partial_{y_1}u\ .$$

The automorphism of A_{m+n}

$$x_j' = x_j \ , \ y_l' = y_l - p_l(x) \ , \ (\partial_{x_j})' = \partial_{x_j} + \sum_l \frac{\partial p_l}{\partial x_j}\partial_{y_l} \ , \ (\partial_{y_l})' = \partial_{y_l}$$

transforms this to $N/\Sigma_l y_l' N$ with the A_m-structure given just by left multiplication. This is holonomic, by (repeated application of) 3.1.7 . It remains to be shown that the automorphism does not affect the holonomicity. This follows from theorem 2.2.2 which gives a filtration free criterion of holonomicity.

Using 3.2.2 we proved

3.2.3 THEOREM. *If* M *is holonomic then* P*M *is holonomic.*

Remark. In general P*M might be not even finitely generated if M is . (E.g. : $M = A_1$, $P : K \to K$, $P(x) = x^2$.)

3.2.4 Define $D_{K^m \to K^n} = P^*A_n$. It carries the left A_m-structure defined above. There is also a right A_n-structure arising from the right A_n-module A_n . With this (A_m, A_n)-bimodule the inverse image P*M can be written in a different way :

$$P^*M = K[x] \otimes_{K[y]} M \cong (K[x] \otimes_{K[y]} A_n) \otimes_{A_n} M = D_{K^m \to K^n} \otimes_{A_n} M \ .$$

We obtain the following formulas.

$$D_{K^{m+n} \xrightarrow{\pi} K^n} \cong A_{m+n} / \sum_{j=1}^{m} A_{m+n}\partial_{x_j} \ ,$$

$$D_{K^m \xrightarrow{i} K^{m+n}} \cong A_{m+n} / \sum_{i=1}^{n} (y_i - p_i(x)) A_{m+n} \ ,$$

$$D_{K^m \xrightarrow{P} K^n} \cong A_{m+n} / \sum_{j=1}^{m} A_{m+n}\partial_{x_j} + \sum_{i=1}^{n} (y_i - p_i(x)) A_{m+n} \ ,$$

with the bimodule structure as above.

3.3 *Direct images*. Again let $P : K^m \to K^n$ be a polynomial mapping, and M a *right* A_m-module. There is a natural definition of a direct image P_*M, this is going to be a right A_n-module, such that for every left A_n-module N there is a canonical isomorphism

$$P_*M \otimes_{A_n} N \cong M \otimes_{A_m} P^*N ,$$

namely $P_*M = M \otimes_{A_m} D_{K^m \to K^n}$. In order to obtain a definition for direct images of left A_n-modules we notice that the transposition

$$(\Sigma_\alpha a_\alpha(x)\partial^\alpha)^t = \Sigma_\alpha(-1)^{|\alpha|}\partial^\alpha a_\alpha(x) \ , \ a_\alpha(x) \in K[x] ,$$

defines an involutive anti-isomorphism of A_m. (I.e. $(D_1D_2)^t = D_2^tD_1^t$.) The application of the transposition to A_m turns a right (left) A_m-module into a left (right) A_m-module.

Remark. This definition of "turning left to right and vice versa" does not directly carry over to the more general case of D-modules on affine varieties. Then the "transposed" modules will in general also be different as modules (over the structure sheaf).

The transpositions of both the A_m- and the A_n-structures of $D_{K^m \to K^n}$ yields the (A_n,A_m)- bimodule $D_{K^n \leftarrow K^m}$ with A_n acting on the left and A_m on the right side.

Denote the transposition of a module M by M^t. If M is now a left A_m-module then its direct image under P is defined by

3.3.1 DEFINITION. $P_*M = D_{K^n \leftarrow K^m} \otimes_{A_m} M \cong (P_*M^t)^t$. ($P_*M^t$ was defined above.)

3.3.2 PROPOSITION. *If* $Q : K^\ell \to K^m$ *is another polynomial mapping, then there is a canonical isomorphism of bimodules.*

$$D_{K^n \leftarrow K^\ell} \cong D_{K^n \leftarrow K^m} \otimes_{A_m} D_{K^m \leftarrow K^l}$$

Proof. This follows from the isomorphism

$$D_{K^1 \to K^n} \cong D_{K^1 \to K^m} \otimes_{A_m} D_{K^m \to K^n} ,$$

which is just proposition 3.2.2 together with 3.2.4 .

3.3.3 PROPOSITION. *If* $P : K^m \to K^n$ *is a polynomial isomorphism, then there are canonical isomorphisms of bimodules*

$$D_{K^m \to K^n} \otimes_{A_n} D_{K^n \leftarrow K^m} \cong A_m ,$$

$$D_{K^n \leftarrow K^m} \otimes_{A_m} D_{K^m \to K^n} \cong A_n ,$$

i.e. P^* *and* P_* *are inverse to each other. (*$m = n$ *of course, but different letters are used in order to avoid confusion.)*

Proof. Identify $D_{K^m \to K^n} = 1 \otimes A_n$ and $D_{K^n \leftarrow K^m}$ with A_n by $1 \otimes D \to D$ and $1 \otimes D \to D^t$, resp. The induced bimodule structures on A_n are

$$A_m \times A_n \times A_n \to A_n \qquad A_n \times A_n \times A_m \to A_n$$

and

$$(D_1, D, D_2) \to (P_* D_1) D D_2 \quad (D_1, D, D_2) \mapsto D_1 D (P_* D_2) .$$

Here $P_* : A_m \to A_n$ is the isomorphism defined by

$$K[x] \ni f \mapsto f \circ P^{-1} , \ \partial_{x_j} \mapsto \Sigma_j \left(\frac{\partial p_i}{\partial x_j} \circ P^{-1}\right) \partial_{y_i} .$$

With this identifications the isomorphisms are

$$D_1 \otimes D_2 \mapsto P_*^{-1}(D_1 D_2) \quad \text{and} \quad D_1 \otimes D_2 \mapsto D_1 D_2 .$$

Let $P : K^m \to K^n$ be a general polynomial mapping. We want to prove that P_* preserves the holonomicity of left A_m-modules. As in 3.2 we split P into an inclusion and a projection : $P = \pi \circ i$, and study two cases seperately (3.3.2).

$\underline{i_*}$: The inclusion is the composition

$$K^m \xrightarrow{j} K^{m+n} \xrightarrow{Q} K^{m+n} \ , \ x \to (x,0), (x,y) \to (x,y+P(x)) \ .$$

As Q is an isomorphism, 3.3.3 implies $Q_* = (Q^{-1})^*$, so by 3.3.2 and 3.2.3 we only have to study the inclusion $j : x \mapsto (x,0)$. From the discussion of i^* we see that

$$D_{K^{m+n} \xleftarrow{i} K^m} \cong A_{m+n}/\sum_{i=1}^{n} A_{m+n}(y_1 - P_1(x)) \ ,$$

$$D_{K^{m+n} \xleftarrow{j} K^m} \cong A_{m+n}/\sum_{i=1}^{n} A_{m+n}y_1 \ .$$

If M is a holonomic left A_m-module, then

$$j_*M \cong A_{m+n}/\Sigma_1 A_{m+n}y_1 \otimes_{A_m} M$$

is a holonomic A_{m+n}-module by repeated application of 3.1.4.2 (use 3.3.2).

$\underline{\pi_*}$: $\pi : K^{m+n} \to K^n$, $(x,y) \mapsto y$. First compute $D_{\leftarrow}$:

$$D_{K^{m+n} \leftarrow K^n} \cong A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i} \quad \text{implies}$$

$$D_{K^n \leftarrow K^{m+n}} \cong A_{m+n}/\sum_{i=1}^{m} \partial_{x_i} A_{m+n} \ .$$

If M is a holonomic left A_{m+n}-module, then we get

$$\pi_*M \cong M/\sum_{i=1}^{m} \partial_{x_i} M \ .$$

This is holonomic over A_n by successive application of 3.1.8 .

3.3.4 THEOREM. *If* M *is a holonomic* A_m*-module then* P_*M *is a holonomic* A_n*-module.*

3.3.5 *Remark.*

$$D_{K^n \xleftarrow{P} K^m} \cong A_{m+n} / \sum_{i=1}^{m} \partial_{x_i} A_{m+n} + \sum_{i=1}^{n} A_{m+n}(y_l - p_l(x)) .$$

Here y_l , ∂_{y_l} operate on the left by multiplication with themselves; on the right x_i operates by multiplication with x_i , ∂_{x_i} by multiplication with $\partial_{x_i} + \Sigma_l \frac{\partial p_l}{\partial x_i} \partial_{y_l}$.

3.4 *Derived inverse images.* Denote by $\mu(A_n)$ the category of left A_n-modules and by $D^-(\mu(A_n))$ and $D^b(\mu(A_n))$ the corresponding derived categories of bounded above and bounded complexes, resp.

Let $P : K^m \to K^n$ be a polynomial mapping. As the functor $P^* : \mu(A_n) \to \mu(A_m)$ is given by the tensor product with the (A_m, A_n)-module $D_{K^m \to K^n}$, there exists a left derived functor $LP^* : D^-(\mu(A_n)) \to D^-(\mu(A_m))$. One way to construct LP^* is to look for a resolution $\tilde{D}^{\bullet}_{K^m \to K^n}$ by bimodules which are projective (or flat or free) as right A_n-modules. A standard free resolution is given by the Koszul complex:

$$\tilde{D}^{\bullet}_{K^m \to K^n} := K_r^{\bullet}(A_{m+n}/\Sigma_i A_{m+n} \partial_{x_i} \; ; y_1 - p_1(x), \ldots, y_n - p_n(x)) .$$

Comments. 1) If M is a right R-module, R some (unitary) ring, and $a_1, \ldots, a_n$ are commuting endomorphisms of M, then the right Koszul complex $K_r^{\bullet}(M; a_1, \ldots, a_n)$ is defined by

$$K_r^{k-n}(M; a_1, \ldots, a_n) := M \otimes_{\mathbb{Z}} \Lambda^k(\mathbb{Z}^n) \ni u \otimes (v_1 \wedge \ldots \wedge v_k)$$

$$\downarrow d \qquad\qquad \downarrow$$

$$M \otimes_{\mathbb{Z}} \Lambda^{k+1}(\mathbb{Z}^n) \ni \Sigma a_j(u) \otimes (e_j \wedge v_1 \wedge \ldots \wedge v_k) .$$

Here e_j is the j-th unit vector.

2) In the case at hand the $y_l - p_l(x)$ operate by multiplication from the left.

Since $[\partial_{x_i} + \Sigma_l \frac{\partial p_l}{\partial x_i}\partial_{y_l}, y_j - p_j(x)] = 0$, left multiplication with $y_j - p_j(x)$ commutes with the left A_m-action. Therefore, the differentials are (A_m, A_n)-linear. It is clear, that the module $A_{m+n}/\Sigma_{i=1}^m A_{m+n}\partial_{x_i}$ is right-A_n-free. The $y_j - p_j(x)$ are a regular sequence for the left $K[x,y]$-module $A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i}$. so the Koszul complex gives a resolution of

$$D_{K^m \to K^n} \cong A_{m+n}/\Sigma_i A_{m+n}\partial_{x_i} + \Sigma_j (y_j - p_j(x))A_{m+n} \, .$$

If $M^\bullet$ is an object of $D^-(\mu(A_n))$ then $LP^*(M^\bullet)$ is given by

$$LP^*(M^\bullet) \cong \tilde{D}^\bullet_{K^m \to K^n} \otimes_{A_n} M^\bullet$$

(On the right hand side one has to take the single complex of a double complex.)

Since $\tilde{D}^\bullet_{K^m \to K^n}$ is bounded, this tensor product also defines a functor $D^b(\mu(A_n)) \to D^b(\mu(A_m))$ which is also denoted by LP^* .

We shall prove two properties of left derived images: First, LP^* depends functorially on P, and secondly, LP^* preserves the holonomicity of the cohomology of objects of $D^-(\mu(A_n))$.

Let $P : K^m \to K^n$ be as above and $Q : K^n \to K^p$ a second polynomial mapping.

3.4.1 PROPOSITION. *On* D^- *and* D^b, $(LP^*) \circ (LQ^*) \cong L(P^* \circ Q^*) (\cong L(Q \circ P)^*)$.

Proof. We have to show that the complexes of (A_m, A_p)-modules

$$\tilde{D}^\bullet_{K^m \to K^p} \text{ and } \tilde{D}^\bullet_{K^m \to K^n} \otimes_{A_n} \tilde{D}^\bullet_{K^n \to K^p}$$

are homology isomorphic. Writing $Q \circ P$ as the composition of the map $K^m \to K^{m+n+p}$ given by $x \mapsto (x, P(X), (Q\circ P)(x))$ and of the projection $(x,y,z) \mapsto z$ of K^{m+n+p} onto K^p, we obtain a new representation and resolution of $D_{K^m \to K^p}$:

$$\hat{D}^{\bullet}_{K^m \to K^p} := K^{\bullet}_{r}\Big(A_{m+n+p}/\sum_i A_{m+n+p}\partial_{x_i} + \sum_j A_{m+n+p}\partial_{y_j};\, y_1 - p_1(x), \dots$$

$$\dots, z_1 - q_1(P(x)), \dots\Big)$$

LEMMA. *The map*

$$[f(x,y)\partial_y^{\alpha}] \otimes [g(y,z)\partial_z^{\beta}] \mapsto [f(x,y)(\partial_y + \frac{\partial Q}{\partial y}\partial_z)^{\alpha}(g(y,z)\partial_z^{\beta})].$$

defines an isomorphism of (A_m, A_p)*-modules*

$$(A_{m+n}/\sum_i A_{m+n}\partial_{x_i}) \otimes_{A_n} (A_{n+p}/\sum_j A_{n+p}\partial_{y_j}) \to A_{m+n+p}/(\sum_i A_{m+n+p}\partial_{x_i} + \sum_j A_{m+n+p}\partial_{y_j}),$$

(Be aware of the structures of these three modules depending on P and Q as explained in 3.2 and 3.3.5 for the $D_{\leftarrow}$-case. On the right hand side, the right A_p-structure is just given by multiplication, as is the left action by x_i . The ∂_{x_i} operate by multiplication with

$$x_i + \sum_j \frac{\partial p_j}{\partial x_i}\partial_{y_j} + \sum_k \frac{\partial (Q\circ P)_k}{\partial x_i}\partial_{z_k} \quad .)$$

The proof of this lemma is similar to that of 3.2.2 , it is essentially the chain rule for $Q\circ P$.

This lemma yields an isomorphism of (A_m, A_p)-complexes

$$\tilde{D}^{\bullet}_{K^m \to K^n} \otimes_{A_n} \tilde{D}^{\bullet}_{K^n \to K^p} \cong \hat{D}^{\bullet}_{K^m \to K^p}$$

proving the proposition.

3.4.2 THEOREM. *Let P be as above and* $M^\bullet$ *an object of* $D^-(\mu(A_n))$ *or* $D^b(\mu(A_n))$ *with holonomic cohomology, i.e. all the* $H^k(M^\bullet)$ *are holonomic. Then* $LP^*(M^\bullet)$ *is also (cohomologically) holonomic.*

The proof of this theorem is similar to the analogous proof (of 3.5.2 below) for direct images in the next section, where a proof is given.

3.5 *Derived direct images.* Let $P : K^m \to K^n$ be a polynomial mapping. Then $P_* : \mu(A_m) \to \mu(A_n)$ is given by the tensor product with $D_{K^n \leftarrow K^m}$, so there is a left derived functor $LP_* : D^-(\mu(A_m)) \to D^-(\mu(A_n))$. A resolution $\tilde{D}^\bullet_{K^n \leftarrow K^m} \to D_{K^n \leftarrow K^m}$ of $D_{K^n \leftarrow K^m}$ (a formula of this is in 3.3.5) by (A_n, A_m)-modules which are free as A_m-modules is

$$\tilde{D}^\bullet_{K^n \leftarrow K^m} := K^\bullet_r(A_{m+n}/\sum_j A_{m+n}(y_j - p_j(x)); \partial_{x_i}, \ldots, \partial_{x_n}) .$$

This gives

$$LP_*(M^\bullet) = \tilde{D}^\bullet_{K^n \leftarrow K^m} \otimes_{A_m} M^\bullet \quad \text{for } M^\bullet D^-(\mu(A_m)) .$$

Since $D^\bullet_{K^n \leftarrow K^m}$ is bounded, we also have a derived functor

$$LP_* : D^b(\mu(A_m)) \to D^b(\mu(A_n)) .$$

Remark. In subsequent chapters we shall see that in the case of a morphism $P : X \to Y$ between non affine varieties there is a direct image functor between the corresponding categories of D-modules. That functor is the composition of first a left derived functor as here and a right derived functor. It will be denoted there RP_+ or P_+ .

Let $Q : K^n \to K^p$ be a second polynomial mapping.

3.5.1 PROPOSITION. $(LQ_*) \circ (LP_*) \cong L(Q_* \circ P_*)$ $(\cong L(Q \circ P)_*)$.

The proposition can be obtained from 3.4.1 by transpositions.

3.5.2 THEOREM. *Let* $M^{\cdot}$ *be an object of* $D^{-}(\mu(A_m))$ *or* $D^{b}(\mu(A_m))$ *with holonomic cohomology. Then* $LP_*(M^{\cdot})$ *has holonomic cohomology.*

Proof. We write $P : K^m \to K^n$ as the composition

$$K^m \xrightarrow{i} K^{m+n} \xrightarrow{\alpha} K^{m+n} \xrightarrow{\pi} K^n$$

$$x \longmapsto (x,0) \qquad (x,y) \longmapsto y$$

$$(x,y) \longmapsto (x,y+P(x))$$

In view of 3.5.1 we can prove the theorem by proving it for i, α, π separately.

$\underline{i}$: $D_{K^{m+n} \xleftarrow{i} K^n} \cong A_{m+n}/\Sigma_j A_{m+n} y_j$ is free as a right A_m-module, so i_* is exact. Since i_* commutes with the cohomology functor, the result follows from 3.3.4 or 3.1.4.2 .

$\underline{\alpha}$: Since α is a isomorphism, α_* is exact. Then use 3.3.4 , for example.

$\underline{\pi}$: One can split π into a sequence of projections with one-dimensional fibres. By 3.5.1 is therefore enough to consider the case $m = 1$. But first, let m be general. Let $N^{\cdot}$ be an object of $D^{-}(\mu(A_{m+n}))$. Then $L\pi_*(N^{\cdot})$ is represented by

$$K_r^{\cdot}(A_{m+n};\partial_{x_i},\dots,\partial_{x_n}) \otimes_{A_{m+n}} N^{\cdot} .$$

In order to prove the theorem we look at one of the spectral sequences of the double complex:

$$K_r^p(A_{m+n};\partial_{x_i},\dots,\partial_{x_n}) \otimes_{A_{m+n}} N^q =: E_0^{p,q} , \quad d_0 : E_0^{p,q} \to E_0^{p,q+q} .$$

Then $E_1^{p,q} \cong K_r^p \otimes_{A_{m+n}} H^q(N^{\cdot})$. Now let $m = 1$, then

$$E_2^{-1,q} \cong \operatorname{Ker}_{H^q(N^\cdot)} \partial_x \ , \ E_2^{0,q} \cong H^q(N^\cdot)/\partial_x H^q(N^\cdot) \ ,$$

$$E_2^{p,q} = 0 \text{ for } p \neq -1,0 \ , \quad E_\infty = E_2 \ .$$

Let us assume the $H^q(N^\cdot)$ to be holonomic. Then $E_2^{-1,q}$ and $E_2^{0,q}$ are A_n-holonomic by 3.1.5 and 3.1.7. E_∞ being holonomic means, there is a filtration of the $H^k(L\pi_*(N^\cdot))$ with holonomic quotients. But then 1.11.1 implies that $H^k(L\pi_*(N^\cdot))$ are holonomic themselves. This proves the theorem.

3.5.3 *Remark.* The complex

$$DR^\cdot(N^\cdot;\partial_{x_i},\ldots,\partial_{x_m}) = DR^\cdot(N^\cdot) = (K_r^\cdot \otimes_{A_{m+n}} N^\cdot)[-m] = N^\cdot \otimes_{\mathbb{Z}} \Lambda^\cdot \mathbb{Z}^m$$

with differential

$$d : \bigoplus_{p+q=k} (N^q \otimes \Lambda^p \mathbb{Z}^m) \to \bigoplus_{p+q=k+1} (N^q \otimes \Lambda^p \mathbb{Z}^m)$$

$$d(u \otimes v) = (-1)^p du \otimes v + \sum_{i=1}^{m} \partial_{x_i} u \otimes \ell_i \wedge v$$

$$(u \in N^q,\ v \in \Lambda^p \mathbb{Z}^m)$$

is called the de Rham complex of $N^\cdot$ relative to π or with respect to the ∂_{x_i} . The last step on the proof of the preceding theorem shows that the relative de Rham complex of an A_{m+n}-complex with holonomic cohomology has A_n-holonomic cohomology groups.

Remark. It has been shown by J. T. Stafford [St] that simple modules over A_n are not in general holonomic and do not share some of the finiteness properties proved here for holonomic ones.

BIBLIOGRAPHICAL NOTE

Most of the material of this chapter is contained in the two papers

[B1] BERNSTEIN, I.N., Modules over the rings of differential operators, a study of the fundamental solutions of equations with constant coefficients, Funct. anal. and Appl. 5,2 (1971), 1-16.

[B2] BERNSTEIN, I.N., The analytic continuation of generalized functions with respect to a parameter, Funct. Anal. and Appl. 6,4 (1972), 26-40.

The notions of §1 are all found in [B1], whereas the central result of §1, Bernstein's inequality, is proved in [B2]. The notions of inverse and direct images under polynomial mappings and the holonomicity theorems of §3 are also part of [B2]. The presentation of the homological algebra results of §2 follows:

[Bj] BJÖRK, J.E., Rings of differential operators, North-Holland Publishing Company, Amsterdam, 1979.

The main results about homological dimension are due to J.-E. Roos:

ROOS, J.E., Détermination de la dimension homologique globale des algèbres de Weyl, C.R. Acad. Sci. Paris **274** (1972), 23-26 .

For examples and properties of simple non-holonomic modules over A_n, see

[St] Stafford, J.T., *Non-holonomic modules over Weyl algebras and enveloping algebras*, Inv. Math. 79(1985), 619-638.