On a theorem of Ranee Brylinski

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1 Introduction

In her thesis [RB], Ranee Brylinski (then Gupta) studied the orbit structure of the projective variety of abelian subalgebras of a fixed dimension, k , in a simple Lie algebra, $\mathfrak{g}$, over $\mathbb{C}$ under its adjoint group, G . Fix a Borel subalgebra, $\mathfrak{b}$, of $\mathfrak{g}$ and let B be the closed subgroup of G corresponding to $\mathfrak{b}$. Then the Borel fixed point theorem implies that the closed G -orbits are precisely the orbits of the abelian ideals in $\mathfrak{b}$ of dimension k . Let l be the rank of $\mathfrak{g}$ and fix $\mathfrak{h}$ a Cartan subalgebra. The main result in [RB] is that the closed orbits are all contained in the closures of the orbits of the subspaces of $\mathfrak{h}$ of dimension k . In particular, if $k = l$ the closure of the orbit of $\mathfrak{h}$ contains all of the closed orbits.

In this paper we will only study the case when $k = l$ (although we expect that similar methods will work for $k < l$) and we prove the following strengthening of Brylinski's theorem.

Theorem 1 *Let x be a regular (not necessarily semisimple) element of $\mathfrak{g}$ contained in $\mathfrak{b}$. Then the closure of the B -orbit of $\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [x, y] = 0\}$ contains all of the abelian ideals of $\mathfrak{b}$ of dimension l .*

Since there is a regular nilpotent element in the closure of the B orbit of each regular element of the projective space on $\mathfrak{g}$, this theorem follows from the special case of $x = e$ a principal (i.e. regular) nilpotent element (see Proposition 3 and its corollary). For this reason we will concentrate on the centralizers of regular nilpotent elements.

In [K3] the first named author proved (in particular) that if we look upon $\wedge^l \mathfrak{g}$ as a $\mathfrak{g}$ -module in the usual way and if C is the Casimir operator corresponding to the Killing form of $\mathfrak{g}$ then its eigenvalues on $\wedge^l \mathfrak{g}$ are all less than or equal to l . Let V_l be the l -eigenspace for C on $\wedge^l \mathfrak{g}$. He also proved that relative to the choice of $\mathfrak{b}$ the highest weight spaces of the l -eigenspace in $\wedge^l \mathfrak{g}$ are exactly the lines $\wedge^l \mathfrak{a}$ with $\mathfrak{a}$ an abelian ideal in $\mathfrak{b}$ of dimension l . In light of this we have the following result as an immediate implication of the above theorem.

Corollary 2 *Let $x \in \mathfrak{g}$ be regular then the $\mathfrak{g}$ -cyclic space of $\wedge^l \mathfrak{g}^x$ in $\wedge^l \mathfrak{g}$ is V_l .*

For x regular semisimple this corollary can be found in [RB].

Unfortunately, the proof of Theorem 1 is by a case by case check through the classification of simple Lie algebras over $\mathbb{C}$ (as was Brylinski's original argument). The hearty reader will find a few general theorems (e.g. Theorem 7) that do not use the classification that should be much more palatable and demonstrate a beautiful underpinning of the results of this paper.

We dedicate this work to our long time friend Veervalli (Raja) Varadarajan in honor of his seventieth birthday.

2 Some observations about $\mathfrak{g}^e$.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Φ denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, let Φ^+ be a system of positive roots and let Δ be the corresponding simple roots. If $\alpha \in \Phi$ then let $\mathfrak{g}_\alpha$ denote the corresponding root space. Let B denote the Killing form of $\mathfrak{g}$. We fix a principal nilpotent element $e \in \mathfrak{g}$ which (after possibly acting by G) we write as

$$e = \sum_{\alpha \in \Delta} X_\alpha$$

with $X_\alpha \neq 0$ for $\alpha \in \Delta$. Let $H \in \mathfrak{h}$ be defined by $\alpha(H) = 1$ for all $\alpha \in \Delta$. Then clearly, $[H, e] = e$. If $x \in \mathfrak{g}$ then $x = u + \sum_{\alpha \in \Phi} x_\alpha$ with $u \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$. We define

$$\text{supp} x = \{\alpha \in \Phi | x_\alpha \neq 0\}.$$

Let G denote the adjoint group of $\mathfrak{g}$ (the identity component of the group of automorphisms) and let $\mathcal{O}(\mathfrak{g})$ denote the polynomial functions on $\mathfrak{g}$ and let $\mathcal{O}(\mathfrak{g})^G$ denote the set of all G -invariant polynomials (i.e. $f(gx) = f(x)$, $x \in \mathfrak{g}$ and $g \in G$). If $x \in \mathfrak{g}$, $f \in \mathcal{O}(\mathfrak{g})$ then we define $\nabla f(x) \in \mathfrak{g}$ by

$$\frac{d}{dt} f(x + ty)|_{t=0} = B(\nabla f(x), y)$$

for all $y \in \mathfrak{g}$. One checks that if $f \in \mathcal{O}(\mathfrak{g})^G$ then

$$g \nabla f(x) = \nabla f(gx), g \in G, x \in \mathfrak{g}.$$

Thus in particular, if $f \in \mathcal{O}(\mathfrak{g})^G$ then $\nabla f(x) \in \mathfrak{g}^x = \{y \in \mathfrak{g} | [x, y] = 0\}$.

Let $\phi_1, \dots, \phi_l$ be a minimal set of homogeneous generators for $\mathcal{O}(\mathfrak{g})^G$ ordered by increasing degree. One has $l = \dim \mathfrak{h}$. Chevalley has shown that $\phi_1, \dots, \phi_l$ are algebraically independent. Such a (partially) ordered set $\phi_1, \dots, \phi_l$ is called a basic set of invariants. The first named author has shown

Theorem 3 *Let $x \in \mathfrak{g}$ then x is a regular element in $\mathfrak{g}$ (i.e. $\dim \mathfrak{g}^x = \dim \mathfrak{h}$) if and only if the elements $\nabla \phi_1(x), \dots, \nabla \phi_l(x)$ are linearly independent.*

This result implies that for a regular element and a choice of $\phi_1, \dots, \phi_l$ the elements $\nabla \phi_1(x), \dots, \nabla \phi_l(x)$ form a basis of $\mathfrak{g}^x$.

If V is a vector space over $\mathbb{C}$ and if $x \in V$ is nonzero let $[x]$ denote the corresponding point in the projective space, $\mathbb{P}(V)$.

Proposition 4 *If $x \in \mathfrak{b}$ is a regular element of $\mathfrak{g}$ then $[e]$ is contained in $\overline{B[x]}$.*

Proof. Let $x = x_s + x_n$ be the Jordan decomposition of x . Then we may assume that $x_s \in \mathfrak{h}$ and

$$x_n = \sum_{\substack{\alpha \in \Delta \\ \alpha(x_s) = 0}} x_\alpha$$

with each of the indicated summands nonzero. Consider the TDS (standard basis of a three dimensional simple Lie algebra) containing to $x_n : u, x_n, y$ with $u \in \mathfrak{h}$ and $[y, x_s] = 0$. Set

$$\mathfrak{v} = \bigoplus_{\substack{\alpha \in \Phi^+ \\ \alpha(x_s) \neq 0}} \mathfrak{g}_\alpha.$$

In [K1] it is proved that there exists a map $z : \mathbb{C}^\times \rightarrow \mathfrak{v}$ such that

$$e^{ad(z(t))} x_s = x_s + \frac{1}{t} e'$$

where

$$e' = \sum_{\substack{\alpha \in \Delta \\ \alpha(x_s) \neq 0}} x_\alpha$$

with all the indicated summands nonzero. Thus

$$e^{adz(t)} e^{-\log(t)adu/2} x = x_s + \frac{1}{t} e' + \frac{1}{t} e^{adz(t)} x_n.$$

We write the term $e^{adz(t)} x_n = x_n + w(t)$ with $w(t) \in [\mathfrak{n}, \mathfrak{n}]$. Thus if $s(t) = e^{adz(t)} e^{-\log(t)adu/2}$. Then

$$ts(t)x = tx_s + e' + x_n + w(t).$$

This implies that

$$s(t)[x] = [tx_s + e' + x_n + w(t)].$$

Let $\{t_j\}$ be a sequence in $(0, 1)$ converging in $(0, 1)$ to 0. Then by taking a subsequence we may assume that

$$\lim_{j \rightarrow \infty} [t_j x_s + e' + x_n + w(t_j)]$$

exists. This limit is of the form $[e' + x_n + w_o]$ with $w_o \in [\mathfrak{n}, \mathfrak{n}]$. $e' + x_n + w_o$ is a principal nilpotent element in $\mathfrak{n}$ so there exists $b \in B$ such that $b(e' + x_n + w_o) = e$. This completes the proof. ■

We note that there is a more direct proof that $\overline{G[x]}$ contains $[e]$. Indeed from [K2] we know that there exists $g \in G$ such that $gx = f + z$ with $z \in \mathfrak{b}$

and $f = \sum_{\alpha \in \Delta} x_{-\alpha}$ with $x_{\alpha} \neq 0$ for $\alpha \in \Delta$. We note that $z = \sum_{j \geq 0} z_j$ with $[H, z_j] = jz_j$. Thus $e^{tadH}gx = e^{-t}f + \sum_{j \geq 0} e^{jt}z_j$. Hence

$$e^t e^{tadH}gx = f + \sum_{j \geq 0} e^{(j+1)t}z_j.$$

This implies that $\lim_{t \rightarrow -\infty} [e^{tadH}gx] = [f]$.

Corollary 5 *If $x \in \mathfrak{b}$ is a regular element of $\mathfrak{g}$ then $\mathfrak{g}^e \in \overline{B\mathfrak{g}^x}$ (here the closure is taken in the Grassmannian of l dimensional subspaces of $\mathfrak{g}$).*

Proof. We use the standard imbedding of the Grassmannian into the projective space $\mathbb{P}(\wedge^l \mathfrak{g})$. Then Theorem 3 implies that $\mathfrak{g}^x$ corresponds to the line $[\nabla\phi_1(x) \wedge \cdots \wedge \nabla\phi_l(x)]$ and so the equivariance assertion above for the maps $\nabla\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ for $\phi \in \mathcal{O}(\mathfrak{g})^G$ and the previous result imply this corollary. ■

If $S \subset \Delta$ then we denote by Φ_S the set $\Phi \cap \sum_{\alpha \in S} \mathbb{Z}\alpha$. Let $\mathfrak{g}_S = \mathfrak{h} + \sum_{\alpha \in \Phi_S} \mathfrak{g}_{\alpha}$. Then $e_S = \sum_{\alpha \in S} X_{\alpha}$ is a principal nilpotent element of $\mathfrak{g}_S$.

Lemma 6 *Let $f \in \mathcal{O}(\mathfrak{g})^G$ be homogeneous of degree j and if $S \subset \Delta$ then $\nabla f(e_S) = (j-1)!(-1)^{j-1}ad(e_S)^{j-1}\nabla f(H)$.*

Proof. We have

$$e^{tad(e_S)}\nabla f(H) = \nabla f(e^{tad(e_S)}H) = \nabla f(H - te_S) = t^{j-1}\nabla f\left(\frac{1}{t}H - e_S\right).$$

Thus

$$t^{1-j}e^{tad(e_S)}\nabla f(H) = \nabla f\left(\frac{1}{t}H - e_S\right).$$

Now take the limit as $t \rightarrow +\infty$ and the Lemma follows. ■

Lemma 7 *If $f \in \mathcal{O}(\mathfrak{g})^G$ is homogeneous and such that $\nabla f(e) \neq 0$ then there exists a basic set of invariants for G containing f .*

Proof. Let j be the degree of f . The content of the assertion is that f is not contained in the subalgebra of $\mathcal{O}(\mathfrak{g})^G$ generated by $\sum_{i < j} \mathcal{O}^i(\mathfrak{g})^G$ ($\mathcal{O}^i(\mathfrak{g})$ the polynomials of degree i on $\mathfrak{g}$). This is obvious since $h(e) = 0$ if $h \in \mathcal{O}^i(\mathfrak{g})^G$ with $i > 0$. ■

Lemma 8 *Let $f \in \mathcal{O}^i(\mathfrak{g})^G$ with $i > 0$. If $\text{supp}\nabla f(e) \cap \Phi_S \neq \emptyset$ then $\phi = f|_{[\mathfrak{g}_S, \mathfrak{g}_S]}$ is contained in a set of basic invariants for $[\mathfrak{g}_S, \mathfrak{g}_S]$ and $\text{supp}\nabla\phi(e_S) = \text{supp}\nabla f(e) \cap \Phi_S$.*

Proof. We note that $\mathfrak{g} = \mathfrak{g}_S \oplus V$ with $V = \sum_{\alpha \notin \Phi_S} \mathfrak{g}_{\alpha}$. This decomposition is $ad_{\mathfrak{g}_S}$ -invariant. Let P be the corresponding projection of $\mathfrak{g}$ onto $\mathfrak{g}_S$. Then $P\nabla f(e) = \sum_{\alpha \in \text{supp}\nabla f(e) \cap \Phi_S} \nabla f(e)_{\alpha} \neq 0$. Up to non-zero scalar multiple $c_i = \frac{(-1)^{i-1}}{(i-1)!}$

$$\nabla f(e) = c_i ad(e)^{i-1} \nabla f(H) = c_i ad(e_S + e_{\Delta-S})^{i-1} \nabla f(H)$$

which is equal to $ad(e_S)^{i-1}\nabla f(H)$ plus a sum of terms that are multiples of expressions of the form $u_1 \cdots u_{i-1}\nabla f(H)$ with at least one $u_i = ad(e_{\Delta-S})$ and the other u_j either $ad(e_S)$ or $ad(e_{\Delta-S})$. Hence we see that $Pu_1 \cdots u_{i-1}\nabla f(H) = 0$ for each of those terms. Thus $0 \neq P\nabla f(e) = c_i ad(e_S)^{i-1}\nabla f(H) = \nabla f(e_S) = \nabla f|_{[\mathfrak{g}_S, \mathfrak{g}_S]}(e_S)$. The result now follows from the previous lemma. ■

We are now ready for our first result about supports.

Theorem 9 *If $\mathfrak{g}$ is simple and not of type D_l , E_6 , or E_7 then*

$$supp(\nabla\phi_i(e)) = \{\alpha \in \Phi^+ | \alpha(H) = \deg \phi_i - 1\}.$$

After we prove this result we will look at the corresponding result for D_l, E_6 and E_7 in each case we will show that there is one exceptional i . We set $\mathcal{O}(\mathfrak{g})_+^G = \{f \in \mathcal{O}(\mathfrak{g})^G | f(0) = 0\}$. Since $e \in \mathcal{N} = \{x \in \mathfrak{g} | \phi(x) = 0, \phi \in \mathcal{O}(\mathfrak{g})_+^G\}$ we see that if $\phi, \psi \in \mathcal{O}(\mathfrak{g})^G$ and if $\phi - \psi \in (\mathcal{O}(\mathfrak{g})_+^G)^2$ then $\nabla\phi(e) = \nabla\psi(e)$. Thus in light of Chevalley's theorem and the fact that under our hypotheses $\deg \phi_i \neq \deg \phi_j$ for $i \neq j$ we need only prove the result for one choice of $\phi_1, \dots, \phi_l$. We will prove the theorem by a case by case check.

A_n : We realize $\mathfrak{g}$ (as usual) as $\{x \in M_{n+1}(\mathbb{C}) | \text{tr} X = 0\}$. Then we may take $e = \sum_{i=1}^n E_{ii+1}$ (one nilpotent Jordan block). We take $\phi_i(x) = \text{tr}(x^i)$. Thus (up to scalar multiple) $\nabla\phi_i(x) = x^{i-1}$. If we use the standard “ ε ” notation for the roots then if j is given then the set $\{\alpha \in \Phi^+ | \alpha(H) = j\}$ is 0 if $j > n$ and is equal to $\{\varepsilon_i - \varepsilon_{i+j} | i = 1, \dots, n+1-j\}$. Obviously,

$$e^j = \sum_{i=1}^{n+1-j} E_{ii+j}.$$

This is the content of the theorem for A_n .

For the B_n and C_n case we will introduce some notation. We take L_n to be the skew diagonal $n \times n$ matrix with ones on the main skew diagonal. That is,

$$L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We realize B_n as the elements $x \in M_{2n+1}(\mathbb{C})$ such that $(x^T$ is the transpose of x)

$$xL_{2n+1} + L_{2n+1}x^T = 0.$$

We set J_n equal to the $2n \times 2n$ block matrix

$$\begin{bmatrix} 0 & L_n \\ -L_n & 0 \end{bmatrix}$$

and we realize C_n as the elements $x \in M_{2n}(\mathbb{C})$ such that

$$xJ_n + J_n x^T = 0.$$

With this realization of $\mathfrak{g} = B_n$ or C_n we choose $\mathfrak{b}$ to be the upper triangular elements of $\mathfrak{g}$ and $\mathfrak{h}$ to be the diagonal elements. We note that if x is a non-zero nilpotent element of $\mathfrak{u} = \mathfrak{sl}(2, \mathbb{C})$ and if $(\sigma_m, \mathbb{C}^m)$ is (up to equivalence) the unique m -dimensional irreducible representation of $\mathfrak{u}$ then $\sigma_m(x)$ is conjugate to a single nilpotent Jordan block hence defines a principle nilpotent element of $\mathfrak{sl}(m, \mathbb{C})$. Also the representation σ_m is orthogonal (has a nondegenerate invariant symmetric form) if m is odd, $m = 2n + 1$ and is symplectic (has a nondegenerate skew symmetric form) if m is even, $m = 2n$. This implies that for the cases B_n or C_n a principal nilpotent element is principal nilpotent for the ambient $\mathfrak{sl}(m, \mathbb{C})$ (m respectively, $2n + 1, 2n$) in our realization. Since we can take $\phi_i(v) = \text{tr}(v^{2i})$ in both of these cases. Hence $\nabla \phi_j(v) = 2jv^{2j-1}$. We see that the assertion for these two cases follows from that for the corresponding A_{m-1} .

In type G_2 there is nothing to prove since $l = 2$ and the assertion of the theorem is true for the invariant of degree 2 (yielding e) and the basic invariant of a highest degree (yielding the highest root).

For the other exceptional cases we will use results in [K1] and [K2] to prove the following theorem that simplifies the problem. Recall that the Coxeter number of $\mathfrak{g}$ is $h = \deg \phi_l$.

Theorem 10 *If the number $\deg \phi_i - 1$ is relatively prime to h then*

$$\text{supp}(\nabla \phi_i(e)) = \{\alpha \in \Phi^+ | \alpha(H) = \deg \phi_i - 1\}.$$

Proof. Set G equal to the adjoint group of $\mathfrak{g}$. Let $\hat{\Delta} = \Delta \cup \{-\beta\}$ with β the highest root in Φ^+ . We set

$$z_o = \sum_{\alpha \in \hat{\Delta}} x_\alpha = e + x_{-\beta}.$$

with $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \hat{\Delta}$ and $x_\alpha \neq 0$. We note that z_o is a cyclic element of $\mathfrak{g}$ in the sense of [K1] and [K2] hence in particular it is a regular semisimple element of $\mathfrak{g}$ and hence

$$\mathfrak{u} = \{x \in \mathfrak{g} | [x, z_o] = 0\}$$

is a Cartan subalgebra. We define $x_o \in \mathfrak{h}$ by $\alpha(x_o) = 1$ for all $\alpha \in \Delta$. We put

$$A = \exp(2\pi i \text{ad}(x_o)/h) \in G.$$

Then if $c = e^{2\pi i/h}$ then since $\beta(x_o) = h - 1$, $Az_o = cz_o$. We also observe that the order of A is h and that A is a regular semisimple element of G hence it is a principal element in the sense of [K1]. In [K1] it is shown that the set

of principal elements of G form one conjugacy class and since it has order h on $\mathfrak{u}$ we see that $A|_{\mathfrak{u}}$ defines a Coxeter element of the Weyl group of $\mathfrak{u}$. Let $\phi_1, \dots, \phi_l$ be a set of homogeneous fundamental G invariant polynomials on $\mathfrak{g}$ with $\deg \phi_i \leq \deg \phi_{i+1}$. Since z_o is regular the elements $u_i = \nabla \phi_i(z_o)$ are linearly independent. Set $m_i = \deg \phi_i - 1$. Observe that $m_l = h - 1$. Also

$$Au_i = A\nabla \phi_i(z_o) = \nabla \phi_i(Az_o) = \nabla \phi_i(cz_o) = c^{m_i}u_i.$$

In particular this implies that

$$u_i = v_i + y_i$$

with $adx_o v_i = m_i v_i$ and $adx_o y_i = (m_i - h)y_i$. We note that

$$e^{tadx_o} \nabla \phi_i(z_o) = \nabla \phi_i(e^{tadx_o}(e + x_{-\beta})) = e^{tm_i} \nabla \phi_i(e + e^{-th}x_{-\beta}).$$

We therefore have

$$v_i = \lim_{t \rightarrow +\infty} e^{-tm_i} e^{tadx_o} u_i = \nabla \phi_i(e).$$

If $\gcd(m_i, h) = 1$ then let $1 \leq q_i \leq h - 1$ be defined by $q_i m_i \equiv 1 \pmod{h}$. Then A^{q_i} has order h and is also regular and so it is principle and its restriction to $\mathfrak{u}$ is also a Coxeter element. This implies that there exists w_i in the normalizer of $\mathfrak{u}$ in G such that

$$w_i A w_i^{-1} = A^{q_i}.$$

Since A^{q_i} is a regular element in the Cartan subgroup of G corresponding to $\mathfrak{h}$ we note that $w_i|_{\mathfrak{h}}$ defines an element of the Weyl group of $\mathfrak{h}$. We note that the above conjugacy implies that $A^{q_i} w_i z_o = c w_i z_o$. Hence $A w_i z_o = c^{m_i} w_i z_o$. Hence

$$w_i \widehat{\Delta} = \{\alpha \in \Phi | \alpha(x_o) = m_i\} \cup \{\alpha \in \Phi | \alpha(x_o) = m_i - h\}.$$

We therefore have

$$v_i = \sum_{\substack{\alpha \in \Phi \\ \alpha(x_o) = m_i}} x_\alpha$$

with $x_\alpha \neq 0$ for all the indices in the sum. This proves the theorem. $\blacksquare$

This result implies Theorem 9 for G_2 , F_4 and E_8 and hence completes its proof.

We now determine the supports for the groups not covered by the theorem.

E_6 in this case the numbers $\deg \phi_i - 1$ are respectively 1, 4, 5, 7, 8, 11 and $\deg \phi_6 = 12$. So we need only check the support of $\nabla \phi_i$ for $i = 2, 5$. For $i = 5$ the set $\{\alpha \in \Phi^+ | \alpha(H) = 8\}$ consists of two roots that are conjugate under the outer involution. If $i = 2$ we argue indirectly as follows. We note that we have an $\mathfrak{sl}_2$ -triple $e, 2H, f$. Also, up to scalar multiple we have

$$ad(e)^4 \nabla \phi_2(H) = \nabla \phi_2(e).$$

We can choose e such that it is invariant under the outer automorphism, θ , gotten from the involution of the Dynkin diagram. We also know that the eigenspace 1 for θ is a C_4 and since C_4 has no invariants of degree 5 we see that the restriction of ϕ_2 to $\mathfrak{h}^\theta$ is 0. This implies that $\theta \nabla \phi_2(H) = -\nabla \phi_2(H)$. Since $\theta e = e$ we see that

$$\theta \nabla \phi_2(e) = -\nabla \phi_2(e).$$

The set $\Phi_4^+ = \{\alpha \in \Phi^+ | \alpha(H) = 4\}$ consists of (in Bourbaki notation):

$$\begin{array}{cccccccccccccccccccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ & & & & & & & 1 & & & & 0 & & & & 1 & & & & & 1 & & \end{array}.$$

The support of $\nabla \phi_2(e)$ is this set with the last element deleted. We will complete the proof of this assertion when we finish our study of the case D_n .

E_7 : In this case we have $\deg \phi_7 = 18$ and the numbers $\deg \phi_i - 1$ are respectively 1, 5, 7, 9, 11, 13, 17. Thus the only case not covered by Theorem 9 is $i = 4$ ($m_4 = 9$). In this case the set $\Phi_9^+ = \{\alpha \in \Phi^+ | \alpha(H) = 9\}$ is

$$\begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 0 \\ & & 1 & & & & & & 1 & & & & & 1 & & & & & & 1 & & 1 & & \end{array}.$$

Let us choose a root vector $X_i \neq 0$ for the i -th root above ($i = 1, \dots, 4$). We note that if $S = \Delta - \{\alpha_7\}$ then $[\mathfrak{g}_S, \mathfrak{g}_S]$ is isomorphic with E_6 . Since E_6 doesn't have a basic invariant of degree 10 we see that the last root cannot be in the support (Lemma 8). If we calculate with $S = \Delta - \{\alpha_1\}$ then we find by a calculation for D_6 that $\nabla \phi_4(e_S) \neq 0$ so the first root is in the support. We also note that Φ_{10}^+ consists of

$$\begin{array}{cccccccccccccccccccc} 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 1 & 0 \\ & & 1 & & & & & 1 & & & & & & 1 & & & & \end{array}.$$

We take $e = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$ with $e_i \neq 0$ in the root space for the i -th simple root in the Bourbaki order. Let $X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4$. Then using the list of elements in Φ_{10}^+ we see that $[e, X]$ is given by

$$a_1[e_1, X_1] + a_2[e_3, X_2] + a_2[e_6, X_2] + a_3[e_5, X_3] + a_4[e_4, X_4] + a_4[e_7, X_4].$$

We are solving for a_1, a_2, a_3, a_4 so that In this expression the only term in the $\begin{array}{cccccc} 1 & 2 & 3 & 2 & 1 & 0 \\ & & 1 & & & \end{array}$ root space is $a_4[e_4, X_4]$ so we must have $a_4 = 0$.

(We have already seen this above.) Since $a_1 \neq 0$ and since $[e_1, X_1]$ is in the $\begin{array}{cccccc} 1 & 1 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{array}$ root space and the only other term in that root space is

$[e_6, X_2]$ we see that $a_2 \neq 0$. But $[e_3, X_2]$ is in the $\begin{array}{cccccc} 1 & 2 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{array}$ root space as is $[e_5, X_3]$ thus we must have $a_3 \neq 0$.

D_n : We realize D_n as the Lie algebra $\mathfrak{g} = \{X \in M_{2n}(\mathbb{C}) | XL_{2n} + L_{2n}X^T = 0\}$. We take $\mathfrak{h}$ to be the diagonal elements in $\mathfrak{g}$ that is the elements

$$h = \begin{bmatrix} h_1 & & & & & & & \\ & h_2 & & & & & & \\ & & \ddots & & & & & \\ & & & h_n & & & & \\ & & & & -h_n & & & \\ & & & & & \ddots & & \\ & & & & & & -h_2 & \\ & & & & & & & -h_1 \end{bmatrix}.$$

The element H then is given by $h_i = n - i$. Also we can take the element e to be

$$e = \begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & & \ddots & 1 & 1 & & \\ & & & & 0 & 0 & -1 & \\ & & & & 0 & -1 & & \\ & & & & & & \ddots & \ddots \\ & & & & & & & 0 & -1 \\ & & & & & & & & 0 \end{bmatrix},$$

that is $\sum_{i=1}^{n-1} (E_{i,i+1} - E_{n+i,n+i+1}) + E_{n-1,n+1} - E_{n,n+2}$. We take as the invariants $\nu_i(X) = \text{tr} X^{2i}$ for $i = 1, 2, \dots, n-1$ and $\nu_n(X) = \text{pfaff}(X)$. A direct calculation shows that $\text{supp} \nabla \nu_i(e) = S_{2i-1} = \{\alpha \in \Phi^+ | \alpha(H) = 2i-1\}$, $1 \leq i \leq n-1$.

Lemma 11 $\text{supp} \nabla \text{pfaff}(e) = \{\varepsilon_1 + \varepsilon_n, \varepsilon_1 - \varepsilon_n\}$.

Proof. We will show that (the blanks should be filled with zeros)

$$\nabla \text{pfaff}(e) = c_n \begin{bmatrix} 0 & 0 & \cdots & 1 & -1 & & & \\ & 0 & \cdots & 0 & 0 & & & \\ & & \ddots & \vdots & \vdots & & & \\ & & & 0 & 0 & 0 & \cdots & 1 \\ & & & & 0 & 0 & & -1 \\ & & & & & \ddots & \ddots & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

with $c_n \neq 0$. Indeed, in this case we take σ to be the permutation matrix corresponding to the transposition $n \leftrightarrow n+1$. Then

$$\zeta : x \longmapsto \sigma x \sigma$$

defines an outer automorphism of $\mathfrak{g}$. One checks that if $\alpha \in \Phi_{n-1}^+ - \{\varepsilon_1 + \varepsilon_n, \varepsilon_1 - \varepsilon_n\}$ then $\zeta|_{\mathfrak{g}_\alpha}$ is multiplication by 1 and ζ interchanges the root spaces corresponding to $\varepsilon_1 + \varepsilon_n$ and $\varepsilon_1 - \varepsilon_n$. Also on $\mathfrak{h}$, ζ acts by $h_i \mapsto h_i$ for $i < n$ and $h_n \mapsto -h_n$. This implies that

$$\text{pfaff}(\zeta X) = -\text{pfaff}(X).$$

Now $e = \zeta e$. Thus $\zeta \nabla \text{pfaff}(e) = -\nabla \text{pfaff}(e)$. The result now follows since $\nabla \text{pfaff}(e) \neq 0$.

We will now finish the discussion of E_6 . We note that the Lie subalgebra that corresponds to the roots that have coefficient 0 with respect to the first simple root is the Levi factor $\mathbb{C} \oplus D_5$. This subalgebra contains the Cartan subalgebra. One checks that the restriction of ϕ_2 to this D_5 is the Pfaffian. The projection of the E_6 “ H ” into this D_5 is the corresponding “ H ”. This implies that the projection of $\nabla \phi_2(H)$ into the D_5 is a nonzero scalar multiple of $\nabla \text{pfaff}(H)$. Since the e for E_6 is E_{α_1} plus the e for D_5 we see that in $\text{ad}(e)^4 \nabla \phi_2(H)$ the terms that don’t involve the first simple root are the ones that appear for D_5 and the Pfaffian. This implies that in the list above the first 2 roots are in the support. The next two are the images of the first two under the outer automorphism. This completes the discussion for E_6 . ■

3 Ideals in $\mathfrak{b}$ of dimension l .

In this section we will give a complete description of these ideals for simple Lie algebras over $\mathbb{C}$. Much of this section can be considered to be a review of results in the MIT thesis of Ranee Brylinski (nee Gupta) [RB]. We maintain the notation of the previous section. We note that since $\mathfrak{h} \subset \mathfrak{b}$ if $\mathfrak{a}$ is an ideal in $\mathfrak{b}$ then $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \cap \mathfrak{a}$. It is easy to see that if $\dim \mathfrak{a} = l$ then $\mathfrak{a} \cap \mathfrak{h} = \{0\}$. This implies that $\mathfrak{a}$ is determined by the roots $\Phi(\mathfrak{a}) = \{\alpha | \mathfrak{g}_\alpha \cap \mathfrak{a} \neq \{0\}\}$. Further a subset $\Lambda \subset \Phi^+$ is a set $\Phi(\mathfrak{a})$ for $\mathfrak{a}$ an ideal in $\mathfrak{b}$ of dimension l if and only if it satisfies the following two conditions

1. $|\Lambda| = l$.
2. If $\alpha \in \Phi^+$, $\beta \in \Lambda$ and if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Lambda$.

Fix Λ satisfying these two conditions. We note that this implies that if ψ is the highest root then $\psi \in \Lambda$. Also since $\psi(H) = h - 1$, we see that if $\alpha \in \Lambda$ then $\alpha(H) \geq h - l$. One can check that 1. and 2. imply that if $\alpha, \beta \in \Lambda$ then if $\mathfrak{g}$ is not of type A or D then $\alpha(H) + \beta(H) \geq 2h - 2l \geq h$ so the corresponding $\mathfrak{a}$ is abelian. For type A_n and D_n we will see that the ideals in $\mathfrak{b}$ of dimension l are abelian as part of our case by case study. We will state this as a lemma and indicate below when the proof is complete.

Lemma 12 *If $\mathfrak{g}$ is simple and if $\mathfrak{a}$ is an ideal in $\mathfrak{b}$ with $\dim \mathfrak{a} = l$ then $\mathfrak{a}$ is abelian.*

We will now give a (case by case) classification of the sets Λ .

Λ_n : In this case Φ^+ is the set of all roots $\alpha_{i,j} = \sum_{k=i}^j \alpha_k$ with $\alpha_1, \dots, \alpha_n$ the simple roots in the usual order and $i \leq j$. Clearly, if $\alpha_{i,j} \in \Lambda$ then $\alpha_{r,s} \in \Lambda$ for $1 \leq r \leq i \leq j \leq n$. From this we see that Λ corresponds to an ideal of $\mathfrak{b}$ contained in $\mathfrak{n}$ if and only if it is of the form

$$\{\alpha_{1,m_1}, \alpha_{1,m_1+1}, \dots, \alpha_{1,n}\} \cup \{\alpha_{2,m_2}, \alpha_{2,m_2+1}, \dots, \alpha_{2,n}\} \cup \dots \cup \{\alpha_{k,m_k}, \alpha_{k,m_k+1}, \dots, \alpha_{k,n}\}$$

with $m_1 \leq m_2 \leq \dots \leq m_k$. If $\dim \mathfrak{a} = n$ then we must have $\sum (n - m_i + 1) = n$. Thus

$$n_1 = n - m_1 + 1 \geq n_2 = n - m_2 + 1 \geq \dots \geq n_k = n - m_k + 1 > 0$$

defines a partition of n . We assert that under this condition $m_1 \geq k$. Indeed, if $m_1 < k$ then $n_1 > n - k + 1$. Since $n_i \geq 1$, for $i = 1, \dots, k$ we would have $n = \sum n_i > n - k + 1 + (k - 1) = n$. 1., 2. and the dimension condition imply that the ideal is abelian in this case. This also implies, in this case, that the number of ideals in $\mathfrak{b}$ of dimension n is $p(n)$ the partition function of n . In the “ ε ” notation for roots the Λ as above are the sets:

$$\begin{aligned} & \{\varepsilon_1 - \varepsilon_{n-n_1+2}, \varepsilon_1 - \varepsilon_{n-n_1+3}, \dots, \varepsilon_1 - \varepsilon_{n+1}\} \cup \\ & \{\varepsilon_2 - \varepsilon_{n-n_2+2}, \varepsilon_2 - \varepsilon_{n-n_2+3}, \dots, \varepsilon_2 - \varepsilon_{n+1}\} \cup \dots \cup \\ & \{\varepsilon_k - \varepsilon_{n-n_k+2}, \varepsilon_k - \varepsilon_{n-n_k+3}, \dots, \varepsilon_k - \varepsilon_{n+1}\} \end{aligned}$$

with $n_1 \geq \dots \geq n_k > 0$ and $\sum n_i = n$.

B_n : We take Δ to be $\{\alpha_1, \dots, \alpha_n\}$ in the usual order. Then the positive roots are given by $\alpha_{i,j} = \sum_{k=i}^j \alpha_k$ for $i \leq j$ and

$$\beta_{i,j} = \sum_{k=i}^j \alpha_k + 2\alpha_{j+1,n}, 1 \leq i \leq j < n.$$

We note that $h = 2n$ so if Λ satisfies 1. and 2. above then $\alpha(H) \geq n$ for all $\alpha \in \Lambda$. Since $\alpha_{i,j}(H) = j + 1 - i$ we see that the only possible element of the form $\alpha_{i,j}$ that can be an element of Λ is $\alpha_{1,n} = \alpha_1 + \dots + \alpha_n$. If Λ contains this element then it must contain $\beta_{1,j}$ for $j = 1, \dots, n - 1$ by condition 2. One checks that the set

$$\{\alpha_{1,n}\} \cup \{\beta_{1,j} | j = 1, \dots, n - 1\}$$

satisfies 1. and 2. We may thus assume that all elements of Λ are of the second form. Here we note that if $\beta_{i,j} \in \Lambda$ then $\beta_{p,q} \in \Lambda$ for all $1 \leq p \leq i \leq q \leq j$ this implies that the other Λ are of the form

$$\{\beta_{1,1}, \beta_{1,2}, \dots, \beta_{1,m_1}\} \cup \{\beta_{2,2}, \dots, \beta_{2,m_2}\} \cup \dots \cup \{\beta_{k,k}, \beta_{k,k+1}, \dots, \beta_{k,m_k}\}$$

with $m_i \geq i$ for $i = 1, \dots, k$ and $m_1 \geq m_2 \geq \dots \geq m_k$ with $\sum (m_i - i + 1) = n$. It is, perhaps, clearer to describe these sets in “ ε ” notation. The first set described above ($\{\alpha_{1,n}\} \cup \{\beta_{1,j} | j = 1, \dots, n - 1\}$) is

$$\{\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n, \varepsilon_1\}.$$

The set for $m_1, \dots, m_k$ is given by

$$\{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_{m_1+1}\} \cup \{\varepsilon_2 + \varepsilon_3, \dots, \varepsilon_2 + \varepsilon_{m_2+1}\} \cup \dots \cup \{\varepsilon_k + \varepsilon_{k+1}, \dots, \varepsilon_k + \varepsilon_{m_k+1}\}.$$

Thus the number of such Λ in this case is the number of partitions of n with distinct parts. We note that as a consequence of Euler's identity the number of partitions of n with distinct parts is the same as the number of partitions, $p_o(n)$, of n with an odd parts (c.f. [A] a discussion of the asymptotics can also be found therein which imply $\sim C_1 e^{C_2 \sqrt{n}}$).

C_n : This case is very similar to the previous one. We use the standard labeling of the simple roots. This time the positive roots are in the form $\alpha_{i,j}$, $1 \leq i, j \leq n$ as before and $\beta_{i,j}$ $1 \leq i \leq j < n$ and

$$\beta_{i,j} = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{n-1} \alpha_k + \alpha_n.$$

Assume that Λ satisfies 1. and 2. As before, if $\alpha \in \Lambda$ then $\alpha(H) \geq n$. Thus the only element of the form α_{ij} that can be in Λ is $\alpha_{1,n}$. The only Λ satisfying 1. and 2. that can contain $\alpha_{1,n}$ is

$$\{\alpha_{1,n}, \beta_{1,n-1}, \beta_{1,n-2}, \dots, \beta_{1,1}\}.$$

The other elements are of the form

$$\{\beta_{1,1}, \beta_{1,2}, \dots, \beta_{1,m_1}\} \cup \{\beta_{2,2}, \dots, \beta_{2,m_2}\} \cup \{\beta_{k,k}, \beta_{k,k+1}, \dots, \beta_{k,m_k}\}$$

with $m_1 \geq m_2 \geq \dots \geq m_k$ and $m_i \geq i$ and $\sum (m_i - i + 1) = n$. We will write these sets in “ ε ” notation. The first is

$$\{\varepsilon_1 + \varepsilon_n, \varepsilon_1 + \varepsilon_{n-1}, \dots, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1\},$$

the others are

$$\{\varepsilon_1 + \varepsilon_{m_1}, \dots, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1\} \cup \{\varepsilon_2 + \varepsilon_{m_2}, \dots, \varepsilon_2 + \varepsilon_3, 2\varepsilon_2\} \cup \dots \cup \{\varepsilon_k + \varepsilon_{m_k}, \dots, \varepsilon_k + \varepsilon_{k+1}, 2\varepsilon_k\}.$$

These all clearly satisfy the conditions 1., 2. Thus number of such ideals in this case is also equal to the number of partitions of n with distinct parts.

D_n : We will only consider the case $n \geq 4$. We note that $h = 2n - 2$. Let Λ satisfy 1. and 2. If $\alpha \in \Lambda$ then the argument above implies that $\alpha(H) \geq n - 2$. If $\alpha(H) = n - 2$ and if there exist two simple roots γ, δ such that $\alpha + \gamma$ and $\alpha + \delta$ are roots then α cannot be a member of Λ (since then $|\Lambda| \geq n + 1$). If n is even then every element at this level has this property so if n is even and if $\alpha \in \Lambda$ then $\alpha(H) \geq n - 1$. If n is odd consider the roots at level $n - 2$. For $n = 2k + 1$ the only α at this level such that we can add exactly one simple root to get a root is

$$\alpha = \alpha_{k+1} + 2\alpha_{k+2} + \dots + 2\alpha_{2k-1} + \alpha_{2k} + \alpha_{2k+1}.$$

To this root we can only add α_k . However to $\alpha + \alpha_k$ we can add two simple roots, α_{k-1} and α_{k+1} and get a root. Thus there are no roots at level $n - 2$ in Λ . This implies that if $\alpha, \beta \in \Lambda$ then $\alpha(H) + \beta(H) \geq h$ so the corresponding ideal in $\mathfrak{b}$ is abelian. This completes the proof of Lemma 12.

It is convenient to continue our discussion of the possible Λ using the “ ε ” notation for the roots thus we identify $\mathfrak{h}$ with $\mathbb{C}^n$ and so $H = (n-1, n-2, \dots, 1, 0)$. The above implies that the only $\alpha \in \Phi^+$ that can possibly be elements of Λ are $\varepsilon_1 - \varepsilon_n$ and $\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$. We observe that if $\varepsilon_i + \varepsilon_j \in \Lambda$ then $\varepsilon_p + \varepsilon_q \in \Lambda$ for $1 \leq p \leq i < q \leq j$. We first assume that $\varepsilon_1 - \varepsilon_n \in \Lambda$ then $\varepsilon_1 + \varepsilon_j \in \Lambda$ for any $2 \leq j \leq n - 1$. Thus there is at most one more root in Λ . If the root were of the form $\varepsilon_i + \varepsilon_j$, $3 \leq i < j \leq n$ then we must have $\varepsilon_{i-1} + \varepsilon_j \in \Lambda$. Similarly if $\varepsilon_2 + \varepsilon_j \in \Lambda$ for $j > 3$ then we would have the same problem. So there would be too many roots in Λ . Thus the element must be of the form $\varepsilon_1 + \varepsilon_j$ or $\varepsilon_2 + \varepsilon_3$. Hence Λ can only be one of

$$\{\varepsilon_1 - \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_n\}$$

or

$$\{\varepsilon_1 - \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_{n-1}\} \cup \{\varepsilon_2 + \varepsilon_3\}.$$

Clearly, both of these sets satisfy 1. and 2. The rest of the possible choices for Λ involve only the elements $\varepsilon_i + \varepsilon_j$. Using the above observation these sets must be of the form

$$\{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_{m_1}\} \cup \{\varepsilon_2 + \varepsilon_3, \dots, \varepsilon_2 + \varepsilon_{m_2}\} \cup \dots \cup \{\varepsilon_k + \varepsilon_{k+1}, \dots, \varepsilon_k + \varepsilon_{m_k}\}.$$

with $m_1 \geq \dots \geq m_k$, $m_i > i$ and the sum $\sum_{i=1}^k (m_i - i) = n$. Again, an easy check says that these sets all satisfy 1. and 2. Here we see that the number of such ideals is one more than the number of partitions of n with distinct parts.

E₆: In this case it is convenient to label the positive roots as 6 tuples where we take the simple roots in the Bourbaki order. One checks that the sets Λ are given as follows:

$$\Lambda_1 = \{(0, 1, 1, 2, 2, 1), (1, 1, 1, 2, 2, 1), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 2, 1)\} \cup \{(1, 1, 2, 3, 2, 1), (1, 2, 2, 3, 2, 1)\},$$

$$\Lambda_2 = \{(1, 1, 2, 2, 1, 0), (1, 1, 1, 2, 2, 1), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 2, 1)\} \cup \{(1, 1, 2, 3, 2, 1), (1, 2, 2, 3, 2, 1)\},$$

$$\Lambda_3 = \{(1, 1, 1, 2, 1, 1), (1, 1, 1, 2, 2, 1), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 2, 1)\} \cup \{(1, 1, 2, 3, 2, 1), (1, 2, 2, 3, 2, 1)\}.$$

E₇: For this case the sets Λ are given as follows. Set

$$V = \{(1, 2, 2, 3, 2, 2, 1), (1, 2, 2, 3, 3, 2, 1), (1, 2, 2, 4, 3, 2, 1)\} \cup \\ \{(1, 2, 3, 4, 3, 2, 1), (2, 2, 3, 4, 3, 2, 1)\}$$

$$\Lambda_1 = \{(1, 2, 2, 3, 2, 1, 0), (1, 2, 2, 3, 2, 1, 1)\} \cup V.$$

$$\Lambda_2 = \{(1, 1, 2, 3, 2, 2, 1), (1, 1, 2, 3, 3, 2, 1)\} \cup V$$

$$\Lambda_3 = \{(1, 2, 2, 3, 2, 1, 1), (1, 1, 2, 3, 3, 2, 1)\} \cup V$$

E₈: Here is the list in this case:

If V is given by

$$\{(2, 3, 3, 5, 4, 3, 2, 1), (2, 3, 4, 5, 4, 3, 2, 1), (2, 3, 4, 6, 4, 3, 2, 1)\} \cup \\ \{(2, 3, 4, 6, 5, 3, 2, 1), (2, 3, 4, 6, 5, 4, 2, 1)\} \cup \\ \{(2, 3, 4, 6, 5, 4, 3, 1), (2, 3, 4, 6, 5, 4, 3, 2)\}$$

then we have

$$\{(1, 3, 3, 5, 4, 3, 2, 1)\} \cup V$$

and

$$\{(2, 2, 4, 5, 4, 3, 2, 1)\} \cup V.$$

F₄: In this case there is only one choice:

$$\{(1, 2, 3, 2), (1, 2, 4, 2), (1, 3, 4, 2), (2, 3, 4, 2)\}.$$

G₂: Here there is only one choice:

$$\{(3, 1), (3, 2)\}.$$

4 The main result.

We retain the notation of the previous sections. We will assume that $\mathfrak{g}$ is simple. Let $G_r(\mathfrak{g})$ denote the Grassmann variety of r -dimensional subspaces of $\mathfrak{g}$ and let $\mathcal{A}_r(\mathfrak{g})$ be the closed subvariety of r -dimensional Lie subalgebras of $\mathfrak{g}$. In this section we will study the natural action of G on $\mathcal{A}_r(\mathfrak{g})$. Fix B a Borel subgroup of G with Lie algebra $\mathfrak{b}$ and $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra (as in the previous sections).

Lemma 13 *Assume that $1 \leq r \leq l$. If $\mathfrak{a} \in \mathcal{A}_r(\mathfrak{g})$ is such that $G\mathfrak{a}$ is closed then there exists a Borel subalgebra $\mathfrak{b}_1$ of $\mathfrak{g}$ such that $\mathfrak{a}$ is an ideal in $\mathfrak{b}_1$ contained in $[\mathfrak{b}_1, \mathfrak{b}_1]$. Furthermore, $\mathfrak{a}$ is abelian.*

Proof. The Borel fixed point theorem implies that B has a fixed point in $G\mathfrak{a}$. Thus by replacing $\mathfrak{a}$ by $g\mathfrak{a}$, if necessary, we may assume that $B\mathfrak{a} = \mathfrak{a}$. Since this implies that $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$ we see that $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{a} \oplus \sum_{\alpha \in \Phi} \mathfrak{a} \cap \mathfrak{g}_\alpha$. This implies that $\mathfrak{a}$ must contain the highest root. Since $\dim \mathfrak{a} = r \leq l$ this implies that $\mathfrak{a} \subset [\mathfrak{b}, \mathfrak{b}]$ and is an ideal in $\mathfrak{b}$. The case by case argument of the previous section implies that $\mathfrak{a}$ is abelian. ■

Our main result is

Theorem 14 *Let $\mathfrak{a}$ be an ideal in $\mathfrak{b}$ of dimension l then $\mathfrak{a}$ is in the Zariski closure of $B\mathfrak{g}^e$.*

This result will be proved by a case by case check. The arguments are complicated but are based on a small number of ideas (most of which appear in the first case). We encourage the serious reader to carry out the steps indicated in the arguments for a specific value of n (for the classical groups). We note that the Borel fixed point theorem implies that the closed orbits of B in $\mathcal{A}_r(\mathfrak{b})$ are fixed points which are therefore the ideals of $\mathfrak{b}$. Thus there is at least one such element in the Zariski closure of $B\mathfrak{g}^e$. This combined with the observations for F_4 and G_2 implies the theorem in these two cases. We now begin the analysis of the other cases.

A_n : We set $e_i = \sum_{j=1}^{n+1-i} E_{jj+i}$ for $i = 1, \dots, n$. Using the standard imbedding of the Grassmannian, $G_l(\mathfrak{b})$ in $\mathbb{P}(\wedge^l \mathfrak{b})$ we must show that if $\mathfrak{a}$ is an abelian ideal in $\mathfrak{b}$ of dimension n then $\wedge^n \mathfrak{a} \in \overline{B(e_1 \wedge \dots \wedge e_n)}$. We will set up a combinatorial method to prove the result in this case. Many of the techniques will be used for the other simple types. We will be using the “ ε ” notation for roots. We will freely interchange partitions, subsets Λ of Φ^+ satisfying 1. and 2. of the previous section and ideals in $\mathfrak{b}$ of dimension l . We need a bit more notation before we get started. Let $H_i = \sum_{j \leq i} E_{ii} - \frac{i}{n+1} I$. We will also use the notation u_i for $E_{ii} - \frac{1}{n+1} I$. Then we note that if $p < q$

$$e^{tadH_i} E_{pq} = \begin{cases} e^t E_{pq}, p \leq i \\ E_{pq}, \text{ otherwise} \end{cases}.$$

The analogous formula for u_i is

$$e^{tadu_i} E_{pq} = \begin{cases} e^t E_{pq}, p = i \\ e^{-t} E_{pq}, q = i \\ E_{pq}, \text{ otherwise} \end{cases}.$$

This implies that

$$e^{tadH_1} e_i = e^t E_{1,i+1} + \sum_{j=2}^{n+1-i} E_{j,j+i}$$

so

$$e^{-tn} e^{tadH_1} (e_1 \wedge \dots \wedge e_n) = E_{12} \wedge E_{13} \wedge \dots \wedge E_{1,n+1} + O(e^{-t})$$

as $t \rightarrow +\infty$. We will use the notation $[w]$ for $\mathbb{C}w$ in $\mathbb{P}(\wedge^l \mathfrak{g})$ (the projective space). The displayed formula now implies that

$$\lim_{t \rightarrow \infty} e^{tadH_1}[e_1 \wedge \cdots \wedge e_n] = [E_{12} \wedge E_{13} \wedge \cdots \wedge E_{1,n+1}]$$

This proves the result for the $\mathfrak{a}$ corresponding to the partition n . For the conjugate partition (a column of n , 1's)

$$\Lambda = \{\varepsilon_1 - \varepsilon_{n+1}, \varepsilon_2 - \varepsilon_{n+1}, \dots, \varepsilon_n - \varepsilon_{n+1}\}.$$

Using H_n we find

$$e^{-tn} e^{tadH_n}(e_1 \wedge \cdots \wedge e_n) = E_{n,n+1} \wedge E_{2,n+1} \wedge \cdots \wedge E_{1,n+1} + O(e^{-t}).$$

Proving the result in this case.

Let $p_1 \geq p_2 \geq \dots \geq p_m$ be a partition of n . We will prove that the corresponding ideal is in the closure of $B \wedge^l \mathfrak{g}^e$. If $m = 1$ or $m = n$ then we have proved the result above. We may assume that $1 < p_1 < n$ and $n > m > 1$. The ideal in $\mathfrak{b}$ corresponding to this partition considered as an element of $\mathbb{P}(\wedge^l \mathfrak{g})$ is

$$[E_{1,n+2-p_1} \wedge \cdots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \cdots \wedge E_{2,n+1} \wedge \cdots \wedge E_{m,n+2-p_m} \wedge \cdots \wedge E_{m,n+1}].$$

We will now approach this element in the closure of $B[e_1 \wedge \cdots \wedge e_n]$ in a sequence of steps. We first show that if $1 \leq k < m$ and $p_k > 1$ then the element

$$[e_{p_1+\dots+p_k-k+1} \wedge \cdots \wedge e_{n-k} \wedge E_{1,n+2-p_1} \wedge \cdots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \cdots \wedge E_{2,n+1} \wedge \cdots \wedge E_{k,n+2-p_k} \wedge \cdots \wedge E_{k,n+1} \wedge E_{k+1,n+1}]$$

is in the closure. We prove this by induction on k . We first note that

$$e^{\frac{1}{t}adE_{1,n+1-p}} e_j = \begin{cases} e_j + \frac{1}{t}E_{1,n+1-(p-j)}, & 1 \leq j \leq p \\ e_j, & p < j \leq n. \end{cases}$$

since $e_n = E_{1,n+1}$ and $e_{n-1} = E_{1,n} + E_{2,n+1}$ we see that if $p > 1$ then

$$t^{p-1} e^{\frac{1}{t}adE_{1,n+1-p}} (e_1 \wedge \cdots \wedge e_n) =$$

$$E_{1,n+2-p} \wedge \cdots \wedge E_{1,n} \wedge e_p \wedge \cdots \wedge e_{n-2} \wedge E_{1,n+1} \wedge E_{2,n+1} + O(t), t \rightarrow 0.$$

Thus we have

$$\lim_{t \rightarrow 0} e^{\frac{1}{t}adE_{1,n+1-p}} [e_1 \wedge \cdots \wedge e_n] = [e_p \wedge \cdots \wedge e_{n-2} \wedge E_{1,n+2-p} \wedge \cdots \wedge E_{1,n} \wedge E_{1,n+1} \wedge E_{2,n+1}].$$

This is the assertion for $k = 1$. Assume that we have proved the assertion for $k-1 \geq 1$ and $k < m$, $p_k > 1$. Then we know that

$$u = [e_{p_1+\dots+p_{k-1}-k+2} \wedge \cdots \wedge e_{n-k+1} \wedge E_{1,n+2-p_1} \wedge \cdots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \cdots \wedge E_{2,n+1} \wedge \cdots \wedge E_{k-1,n+2-p_{k-1}} \wedge \cdots \wedge E_{k-1,n+1} \wedge E_{k,n+1}]$$

is in the closure. We also note that $e_{p_1+\dots+p_{k-1}-k+2} = \sum E_{r,p_1+\dots+p_{k-1}-k+2}$ over $1 \leq r \leq n-1+k-p_1-\dots-p_{k-1}$. We consider the term with $r+p_1+\dots+p_{k-1}-k+2 = n+2-p_k$. That is $r = n-p_1-\dots-p_k+k = p_{k+1}+\dots+p_m+k \geq k+1$ since $m > k$. We will now analyze $e^{\frac{1}{t}adE_{k,r}}u$ as $t \rightarrow 0$. The factors in u that have a term of the form $E_{r,j}$ are $e_{p_1+\dots+p_{k-1}-k+2}, \dots, e_{p_1+\dots+p_{k-1}-k+1+p_k}$ and the terms are $E_{r,n+2-p_k}, \dots, E_{r,n+1}$. As above we have all of the factors in u are fixed by $e^{\frac{1}{t}adE_{k,r}}$ except for these (we note that every term appearing in the product has second entry at least k) and for these

$$e^{\frac{1}{t}adE_{k,r}}e_{p_1+\dots+p_{k-1}-k+1+j} = e_{p_1+\dots+p_{k-1}-k+1+j} + \frac{1}{t}E_{k,n-p_k+1+j}.$$

Now $E_{k,n+1}$ is already a factor see that $\lim_{t \rightarrow 0} e^{\frac{1}{t}adE_{k,r}}u$ is

$$[e_{p_1+\dots+p_k-k+1} \wedge \dots \wedge e_{n-k} \wedge E_{1,n+2-p_1} \wedge \dots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \dots \wedge E_{2,n+1} \wedge \dots \wedge E_{k,n+2-p_k} \wedge \dots \wedge E_{k,n+1} \wedge E_{k+1,n+1}].$$

This completes the induction. We can now prove the theorem in the case of A_n .

If $m = 1$ all of the p_i are 1 then we are done. We may thus assume that $m \geq 2$ and either $p_1 \geq \dots \geq p_m > 1$ or we have $p_1 \geq \dots \geq p_k > 1$ and $p_{k+1} = \dots = p_m = 1$. We apply our assertion above for $k = m - 1$. We then have

$$v = [e_{p_1+\dots+p_{m-1}-m+2} \wedge \dots \wedge e_{n-m+1} \wedge E_{1,n+2-p_1} \wedge \dots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \dots \wedge E_{2,n+1} \wedge \dots \wedge E_{m-1,n+2-p_{m-1}} \wedge \dots \wedge E_{m-1,n+1} \wedge E_{m,n+1}]$$

in the closure. We note that $m + p_1 + \dots + p_{m-1} - m + 2 = n + 2 - p_m$. Thus we have in the first case $\lim_{t \rightarrow +\infty} e^{tadH_m}v =$

$$[E_{1,n+2-p_1} \wedge \dots \wedge E_{1,n+1} \wedge E_{2,n+2-p_2} \wedge \dots \wedge E_{2,n+1} \wedge \dots \wedge E_{m,n+2-p_m} \wedge \dots \wedge E_{m,n+1}]$$

as desired. In the second case we note that if $k = m - 1$ we are already done. If not we argue as in the beginning of the proof to see that

$$\lim_{t \rightarrow \infty} e^{tH_n}v$$

is the desired element. Since we have covered both cases the proof for A_n is complete.

$B_n, n \geq 2$. We use the realization in section 2. We will do this case in a great deal of detail and just show how one modifies the methods for C_n and D_n . We note that if $n = 2$ then there is only one such ideal so we may assume that $n \geq 3$. We assume that the reader has looked at the methods for the previous case. We write

$$X_{\varepsilon_j} = E_{j,n+1} - E_{n+1,n+1+j},$$

$$X_{\varepsilon_i - \varepsilon_j} = E_{ij} - E_{2n+2-j,2n+2-i}, i < j,$$

$$X_{\varepsilon_i + \varepsilon_j} = E_{i, 2n+2-j} - E_{j, 2n+2-i}, i < j.$$

The subsets Λ under consideration are

$$\{\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n, \varepsilon_1\}$$

and

$$\{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_{m_1}\} \cup \{\varepsilon_2 + \varepsilon_3, \dots, \varepsilon_2 + \varepsilon_{m_2}\} \cup \dots \cup \{\varepsilon_k + \varepsilon_{k+1}, \dots, \varepsilon_k + \varepsilon_{m_k}\}$$

for $m_1 \geq m_2 \geq \dots \geq m_k$ with $m_i > i$ and $\sum_i (m_i - i) = n$.

We will also use the notation u_j for the element of $\mathfrak{h}$ such that $\varepsilon_i(u_j) = \delta_{ij}$.

We must show that if $\mathfrak{a}_\Lambda$ is the ideal corresponding to Λ then it is in the closure of $B\mathfrak{g}^e$. We first look at the first of the sets corresponding to ideals in $\mathfrak{b}$. If $n = 2k$ then

$$v_0 = \lim_{t \rightarrow +\infty} e^{tadu_1} [e_1 \wedge \dots \wedge e_n] = \\ [X_{\varepsilon_1 - \varepsilon_2} \wedge \dots \wedge X_{\varepsilon_1 - \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k-2}} \wedge \dots \wedge X_{\varepsilon_1 + \varepsilon_2}].$$

Now

$$v_1 = \lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{\varepsilon_2 + \varepsilon_3}} v_0 = [X_{\varepsilon_1 + \varepsilon_3} \wedge X_{\varepsilon_1 + \varepsilon_4} \wedge \dots \wedge X_{\varepsilon_1 - \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k-2}} \wedge \dots \wedge X_{\varepsilon_1 + \varepsilon_2}]$$

and if v_j has been defined with $j < k - 1$ then

$$v_{j+1} = \lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{\varepsilon_{2j+3} + \varepsilon_{2j-2}}} v_j.$$

We have (recall $n = 2k$)

$$v_k = [X_{\varepsilon_1 - \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k}} \wedge X_{\varepsilon_1 + \varepsilon_{2k-1}} \wedge \dots \wedge X_{\varepsilon_1 + \varepsilon_3} \wedge X_{\varepsilon_1 + \varepsilon_2}]$$

finally

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_n} v_k$$

is the desired element.

The case when $n = 2k + 1$ is almost the same define $v_0, \dots, v_k$ in exactly the same way then v_k is the desired element.

For the rest of the argument in this case it will be convenient to use the notation

$$X_{(i,0)} = X_{\varepsilon_i}, X_{(i,-j)} = X_{\varepsilon_i - \varepsilon_j}, i < j \text{ and } X_{(i,j)} = X_{\varepsilon_i + \varepsilon_j}.$$

Before we continue we will give another description of the sets $\Lambda \subset \Phi^+$ that satisfy condition 2. in section 3. We note that the set must contain a maximal subset of the form $\Lambda^{(r)} = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq r \text{ and } i+1 \leq j \leq r+1\}$. If $\Lambda \neq \Lambda^{(r)}$ then the rest of Λ is given as $\cup_{j=1}^{m_1-r-1} \{\varepsilon_i + \varepsilon_{j+r+1} \mid 1 \leq i \leq p_j\}$ and

$$r \geq p_1 \geq p_2 \geq \dots \geq p_{m_1-r-1} \geq 0$$

satisfies $\sum_j p_j = |\Lambda| - \frac{r(r+1)}{2}$. To see this decomposition we give an example In the array below the first row indicates the elements $\varepsilon_1 + \varepsilon_i$ in Λ second indicates the elements $\varepsilon_2 + \varepsilon_i$ etc.

$$\begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & 3 & 4 & 5 & 6 & 7 & 8 \\ & & 4 & 5 & 6 & 7 & \\ & & & 5 & 6 & & \\ & & & & 6 & & \end{array}$$

So here $m_1 = 8, m_2 = 8, m_3 = 7, m_4 = 6, m_5 = 6, r = 5$ and $\Lambda^{(5)}$ corresponds to

$$\begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 \\ & 3 & 4 & 5 & 6 \\ & & 4 & 5 & 6 \\ & & & 5 & 6 \\ & & & & 6 \end{array}$$

and the rest is

$$\{\varepsilon_1 + \varepsilon_7, \varepsilon_2 + \varepsilon_7, \varepsilon_3 + \varepsilon_7\} \cup \{\varepsilon_1 + \varepsilon_8, \varepsilon_2 + \varepsilon_8\}.$$

So in this case $p_1 = 3, p_2 = 2, p_3 = 0$. Our strategy in the proof is to use this decomposition of the set and handle the two parts individually.

We prove by induction on r that if $\mathfrak{a}^{(r)}$ is the ideal corresponding to $\Lambda^{(r)}$ then if r is odd the subspace corresponding to

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge e_{\binom{r+1}{2}+1-\frac{r+1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}$$

is in the closure of $B\mathfrak{g}^e$ and if r is even and if $n > \binom{r+1}{2}$ then the subspace corresponding to

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge X_{(1,r+2)} \wedge e_{\binom{r+1}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{n-\frac{r+2}{2}}$$

is in the closure of $B\mathfrak{g}^e$ or if r is even and $n = \binom{r+1}{2}$ then $\mathfrak{a}^{(r)}$ is in the closure of $B\mathfrak{g}^e$. The latter case (with the extra $X_{(1,r+2)}$ will be called a “carry”. This is because $e_{n-\frac{r}{2}} \wedge \wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} = X_{(1,r+2)} \wedge \wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)}$.

If $r = 1$ then since e_n is a multiple of $X_{(1,2)}$ and $\mathfrak{a}^{(1)} = \mathbb{C}X_{(1,2)}$ the assertion is that $\mathfrak{g}^e$ is in the closure of $B\mathfrak{g}^e$. If $r = 2$ then the case of equality is $n = 3$. In this case

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{23}} [e_1 \wedge e_2 \wedge e_3] = [e_2 \wedge X_{(1,3)} \wedge X_{(1,2)}]$$

and e_2 is up to nonzero multiple $X_{(1,0)} + cX_{(2,3)}$ with $c \neq 0$. Thus

$$\lim_{t \rightarrow +\infty} e^{t ad u_2} [e_2 \wedge X_{(1,3)} \wedge X_{(1,2)}] = \wedge^3 \mathfrak{a}^{(2)}.$$

Note that this completes the proof of the main result for $n = 3$. If $n > 3$ then

$$v = \lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(2,3)}} [e_1 \wedge e_2 \wedge \cdots \wedge e_n] = [e_2 \wedge \cdots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)}].$$

If $n = 4$ then

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_3} v = [X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}]$$

proving the assertion and completing the proof for $n = 4$. If $n > 4$ then

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(3,5)}} v = [(X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}) \wedge X_{(1,4)} \wedge e_3 \wedge \cdots \wedge e_{n-2}].$$

This completes the initiation of the induction and the proof of the main theorem for $n \leq 5$ we can thus assume that $n \geq 6$.

To go on with the induction we will need a bit more notation. We set

$$\phi_n(r) = \begin{cases} -r & \text{if } 1 \leq r \leq n \\ 0 & \text{if } r = n+1 \\ 2n+2-r & \text{if } n+2 \leq r \leq 2n \end{cases}.$$

We consider the array, A , with $a_{i,j} = (i, \phi_n(2j+i-1))$ for $1 \leq i \leq n-j+1$ and $1 \leq j \leq n-i+1$. For example, if $n = 9$

$$A = \begin{array}{cccccccc} (1, -2) & (1, -4) & (1, -6)) & (1, -8) & (1, 0) & (1, 8) & (1, 6) & (1, 4) & (1, 2) \\ (2, -3) & (2, -5) & (2, -7) & (2, -9) & (2, 9) & (2, 7) & (2, 5) & (2, 3) & \\ (3, -4) & (3, -6) & (3, -8) & (3, 0) & (3, 8) & (3, 6) & (3, 4) & & \\ (4, -5) & (4, -7) & (4, -9) & (4, 9) & (4, 7) & (4, 5) & & & \\ (5, -6) & (5, -8) & (5, 0) & (5, 8) & (5, 6) & & & & \\ (6, -7) & (6, -9) & (6, 9) & (6, 7) & & & & & \\ (7, -8) & (7, 0) & (7, 8) & & & & & & \\ (8, -9) & (8, 9) & & & & & & & \\ (9, 0) & & & & & & & & \end{array}.$$

If we write $\varepsilon(i, m) = \varepsilon_i + \text{sgn}(m)\varepsilon_{|m|}$. (defining $\text{sgn}(0) = 0$). Then the j^{th} column of A is the support of e_j . We will illustrate the method for this case. Looking at the array we see that

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(1,4)}} [e_1 \wedge \cdots \wedge e_9] = [X_{(1,2)} \wedge X_{(1,3)} \wedge e_2 \wedge \cdots \wedge e_8].$$

If we do the same operation using $X_{(2,6)}$ to the outcome we have

$$[X_{(1,2)} \wedge X_{(1,3)} \wedge X_{(2,3)} \wedge e_3 \wedge \cdots \wedge e_8] = [X_{(1,2)} \wedge X_{(1,3)} \wedge X_{(1,4)} \wedge X_{(2,3)} \wedge e_3 \wedge \cdots \wedge e_7] = \\ \wedge^3 \mathfrak{a}^{(2)} \wedge X_{(1,4)} \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7.$$

We now “add” the third row of $\mathfrak{a}^{(3)}$. We “need” $(2, 4)$ and $(3, 4)$. We use $X_{(2,9)}$ (as above) getting

$$\wedge^3 \mathfrak{a}^{(2)} \wedge X_{(1,4)} \wedge X_{(2,4)} \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7$$

and then $X_{(3,-9)}$ to get

$$\wedge^3 \mathfrak{a}^{(2)} \wedge X_{(1,4)} \wedge X_{(2,4)} \wedge X_{(3,4)} \wedge e_5 \wedge e_6 \wedge e_7 = \wedge^6 \mathfrak{a}^{(3)} \wedge e_5 \wedge e_6 \wedge e_7.$$

We now assume that we have proved the assertion for $r - 1$. We first look at the case when $\binom{r+1}{2} = n$. If r is odd then $r - 1$ is even hence the inductive hypothesis implies that $\wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge e_{\binom{r}{2}+1-\frac{r-1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}$ is in the closure of $B\mathfrak{g}^e$. We note that under our hypothesis we have

$$\binom{r}{2} = n - r.$$

Thus if $j = \binom{r}{2} + 1 - \frac{r-1}{2}$ then $\phi_n(2j + (r+1) - 1) = \phi_n(2(n-r+1) - r + 1 + r) = \phi_n(2n - 2r + 3)$. If $2n - 2r + 3 \leq n + 1$ then we would have $\binom{r+1}{2} \leq 2r - 2$ and this is not possible. Thus $\phi_n(2n - 2r + 3) = 2r - 1$. We also note that the element in the third row and the $n - \frac{r+1}{2}$ column is $(3, r+1)$. This implies that the subspace corresponding to

$$\begin{aligned} & \lim_{t \rightarrow +\infty} e^{tadu_{r+1}} \wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge e_{\binom{r}{2}+1-\frac{r-1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}} = \\ & \wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge X_{(r+1,2r-1)} \wedge X_{(r+1,2r-3)} \wedge \cdots \wedge X_{(r+1,r+2)} \wedge \\ & X_{(r,r+1)} \wedge X_{(r-2,r+1)} \wedge \cdots \wedge X_{(3,r+1)}. \end{aligned}$$

Now do in order the limits as $t \rightarrow 0$ of $e^{\frac{1}{t}adX_{(2,-2r+1)}}$, $e^{\frac{1}{t}adX_{(4,-2r+3)}}$, ..., $e^{\frac{1}{t}adX_{(r-1,-r-2)}}$ (here the limit of the left most term is taken first) and we have $\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)}$ as desired.

If r is even and $\binom{r+1}{2} = n$. then the inductive hypothesis implies that the subspace corresponding to

$$\wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge e_{\binom{r}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{n-\frac{r}{2}}$$

is in the closure of $B\mathfrak{g}^e$. If we set $j = \binom{r}{2} + 1 - \frac{r}{2}$ then we see that if $2j + r \leq n + 1$ then $r \leq 2$ but then $n = 3$ and the result has been proved in this case. Thus $\phi_n(2j + (r+1) - 1) = \phi_n(2n - 2r + 2) = 2r$. This time the element in the second row and the $n - \frac{r}{2}$ column is $(2, r+1)$. This implies that

$$\begin{aligned} & \lim_{t \rightarrow +\infty} e^{tadu_{r+1}} \wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge e_{\binom{r}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{n-\frac{r}{2}} = \\ & \wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(r+1,2r)} \wedge X_{(r+1,2r-2)} \wedge \cdots \wedge X_{(r+1,r+2)} \wedge \\ & X_{(r,r+1)} \wedge X_{(r-2,r+1)} \wedge \cdots \wedge X_{(2,r+1)}. \end{aligned}$$

Now argue in the same way as the previous case.

We are now ready to do the general inductive step. We first look at the case when n is sufficiently large compared with $\binom{r+1}{2}$. For us this will mean that if r

is odd then $n \geq \binom{r+1}{2} + \frac{r-1}{2} + 1$ or if r is even $n \geq \binom{r+1}{2} + \frac{r}{2} + 1$. Under this condition we have that if r is odd and $m = \binom{r}{2} - \frac{r-1}{2}$ then $n - r + 1 \geq m + r - 1$. This implies that we can apply in order (left to right)

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(2, \phi_n(2(m+1)+r))}}, \lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(2, \phi_n(2(m+2)+r))}}, \dots, \lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(r, \phi_n(2(m+r-1)+r))}}$$

to get the desired element. If r is even and if $m = \binom{r}{2} - \frac{r}{2}$ then our hypothesis implies that $n - r + 1 \geq m + r$ and so we can apply

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(1, \phi_n(2(m+1)+r))}}, \lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(2, \phi_n(2(m+2)+r))}}, \dots, \lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(r, \phi_n(2(m+r)+r))}}$$

we note that after the second limit we have

$$\wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(1, r+1)} \wedge X_{(2, r+1)} \wedge e_{\binom{r}{2}+3-\frac{r}{2}} \wedge \dots \wedge e_{n-\frac{r}{2}}$$

but $\wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(2, r+1)} \wedge e_{n-\frac{r}{2}} = \wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(2, r+1)} \wedge X_{(1, r+2)}$. We have finally completed this induction.

To complete the proof we recall that if Λ satisfies condition 2 in section 3 then it is of the form

$$\Lambda^{(r)} \cup C_{r+1, p_1} \cup \dots \cup C_{m_1-1, p_{m_1-r-1}}$$

with $C_{k, p} = \{\varepsilon_i + \varepsilon_{k+1} | i = 1, \dots, p\}$ (as above). Let s be the maximum of the even i such that $p_i \geq 2, p_{i-1} \geq 3, \dots$. If no such exists we set $s = 0$. We set $X_C = X_{(a_1, b_1)} \wedge \dots \wedge X_{(a_s, b_s)}$ if $C = \{(a_1, b_1), \dots, (a_s, b_s)\}$. We argue as above that if i is odd and r is odd then if $i < s$ then the subspace corresponding to

$$\wedge \binom{r+1}{2} \mathfrak{a}^{(r)} \wedge X_{C_{r+1, p_1}} \wedge \dots \wedge X_{C_{i, p_i}} \wedge e_{\binom{r+1}{2}+1-\frac{i+1}{2}+p_1+\dots+p_i} \wedge \dots \wedge e_{n-\frac{i+1}{2}}$$

is in the closure of $B\mathfrak{g}^e$, if $i > s$ then the subspace corresponding to

$$\wedge \binom{r+1}{2} \mathfrak{a}^{(r)} \wedge X_{C_{r+1, p_1}} \wedge \dots \wedge X_{C_{i, p_i}} \wedge e_{\binom{r+1}{2}+1-\frac{s}{2}+p_1+\dots+p_i} \wedge \dots \wedge e_{n-\frac{s}{2}}$$

is in the closure, if i is even, $i < m_1 - 1$ and $i \leq s$ then there is a carry that is the subspace corresponding to

$$\wedge \binom{r+1}{2} \mathfrak{a}^{(r)} \wedge X_{C_{r+1, p_1}} \wedge \dots \wedge X_{C_{i, p_i}} \wedge X_{(1, i+1)} \wedge e_{\binom{r+1}{2}+1-\frac{i}{2}-p_1-\dots-p_r} \wedge \dots \wedge e_{n-\frac{i+2}{2}}$$

is in the closure of $B\mathfrak{g}^e$ otherwise there is no carry and

$$\wedge \binom{r+1}{2} \mathfrak{a}^{(r)} \wedge X_{C_{r+1, p_1}} \wedge \dots \wedge X_{C_{i, p_i}} \wedge e_{\binom{r+1}{2}+1-\frac{s}{2}-p_1-\dots-p_r} \wedge \dots \wedge e_{n-\frac{s}{2}}.$$

This (finally) implies the result.

We illustrate this with the example $n = 9, r = 3$ and $p_1 = 2, p_2 = 1$. We have $s = 4$ and

$$\wedge^6 \mathfrak{a}^{(3)} \wedge e_5 \wedge e_6 \wedge e_7$$

is in the closure. Apply $X_{(1,-6)}$ and get

$$\wedge^6 \mathbf{a}^{(3)} \wedge X_{(1,5)} \wedge e_6 \wedge e_7$$

to this apply $X_{(1,-4)}$ and get (the carry)

$$\wedge^6 \mathbf{a}^{(3)} \wedge X_{(1,5)} \wedge X_{(2,5)} \wedge X_{(1,6)}.$$

C_n , $n \geq 3$. We will use the “ ε ” notation for the roots. We will write $\varepsilon(i, j) = \varepsilon_i + \text{sgn}(j)\varepsilon_{|j|}$. Here the positive roots are $\varepsilon_i - \varepsilon_j$ for $i < j$ and $\varepsilon_i + \varepsilon_j$ for $i \leq j$. Thus $\varepsilon(2, 2) = 2\varepsilon_2$. As in the previous case we choose a nonzero root vector, X_α , for each positive root α . We will write $X_{(i,j)} = X_{\varepsilon(i,j)}$ and freely interchange the roots and the pairs. For this case we define

$$\phi_n(r) = \begin{cases} -r, & r \leq n \\ 2n+1-r, & r > n \end{cases}.$$

Then the support of e_j is the set $\{\varepsilon(i, \phi_n(2j+i-1)) | 1 \leq i \leq n+1-j\}$. We lay out the support in an array A where the j -th column is the support of e_j . Thus if $n = 9$ we have

$$A = \begin{array}{cccccccc} (1, -2) & (1, -4) & (1, -6)) & (1, -8) & (1, 9) & (1, 7) & (1, 5) & (1, 3) & (1, 1) \\ (2, -3) & (2, -5) & (2, -7) & (2, -9) & (2, 8) & (2, 6) & (2, 4) & (2, 2) & \\ (3, -4) & (3, -6) & (3, -8) & (3, 9) & (3, 7) & (3, 5) & (3, 3) & & \\ (4, -5) & (4, -7) & (4, -9) & (4, 8) & (4, 6) & (4, 4) & & & \\ (5, -6) & (5, -8) & (5, 9) & (5, 7) & (5, 5) & & & & \\ (6, -7) & (6, -9) & (6, 8) & (6, 6) & & & & & \\ (7, -8) & (7, 9) & (7, 7) & & & & & & \\ (8, -9) & (8, 8) & & & & & & & \\ (9, 9) & & & & & & & & \end{array}.$$

As above we define $u_j \in \mathfrak{h}$ by $\varepsilon_i(u_j) = \delta_{ij}$.

We first look at the set $\Lambda = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_n\}$. Then if $n = 2k$ or $n = 2k + 1$

$$\lim_{t \rightarrow +\infty} e^{tadu_1} [e_1 \wedge \dots \wedge e_n] =$$

$$[X_{(1,-2)} \wedge X_{(1,-4)} \wedge \dots \wedge X_{(1,-2k)} \wedge X_{(1,2k-(-1)^n)} \wedge X_{(1,2k-(-1)^{n-2})} \wedge \dots \wedge X_{(1,3)} \wedge X_{(1,1)}.$$

If we apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} ad \sum_{j=1}^k X_{(2k, 2k)}}$ to this we have $\wedge^n \mathbf{a}_\Lambda$. For the set Λ one can apply the same method of constructing the set Λ column by column as in the case of B_n to prove the main theorem in this case. The arguments are a bit simpler. We won't go into the details but will set up for the calculations. In this case if

$$\Lambda^{(r)} = \{(i, j) | i \leq r, j \leq r, i \leq j\}$$

and $\mathbf{a}^{(r)} = \mathbf{a}_{\Lambda^{(r)}}$ then if $\Lambda \subset \Phi^+$ is the set of roots in an ideal in $\mathfrak{b}$ then

$$\Lambda = \Lambda^{(r)} \cup (\{(1, r+1), (2, r+1), \dots, (p_{r+1}, r+1)\} \cup \dots \cup \{(1, r+k), (2, r+k), \dots, (p_{r+k}, r+k)\})$$

with $r \geq p_{r+1} \geq \dots \geq p_{r+k} \geq 1$. We will show how the procedure works (we will give details only for the case of $n = 9$) first we consider the $\Lambda^{(r)}$. We have if $n \geq 5$ then

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(1,3)}} [e_1 \wedge \dots \wedge e_n] = [e_2 \wedge \dots \wedge e_{n-1} \wedge X_{(1,2)} \wedge X_{(1,1)}].$$

If we apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(2,5)}}$ to this we have

$$[e_3 \wedge \dots \wedge e_{n-1} \wedge X_{(1,2)} \wedge X_{(1,1)} \wedge X_{(2,2)}]$$

we note that

$$[e_{n-1} \wedge X_{(1,2)} \wedge X_{(1,1)} \wedge X_{(2,2)}] = [X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(1,1)} \wedge X_{(2,2)}],$$

a “carry”. We thus have

$$\wedge^{(3)}_{(2)} \mathfrak{a}^{(2)} \wedge X_{(1,3)} \wedge e_3 \wedge \dots \wedge e_{n-2}.$$

We now assume that $n \geq 9$ then applying $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(2,8)}}$ we have

$$\wedge^{(3)}_{(2)} \mathfrak{a}^{(2)} \wedge X_{(1,3)} \wedge X_{(2,3)} \wedge e_4 \wedge \dots \wedge e_{n-2}.$$

Now we assume that $n = 9$ (if $n > 9$ then the next step would involve $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(3,10)}}$) we apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(3,-9)}}$ to this and we get

$$\wedge^{(3)}_{(2)} \mathfrak{a}^{(2)} \wedge X_{(1,3)} \wedge X_{(2,3)} \wedge X_{(3,3)} \wedge e_5 \wedge e_6 \wedge e_7 = \wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_5 \wedge e_6 \wedge e_7.$$

The possible Λ containing Λ_2 correspond to $p_4 = 3; p_4 = 2, p_5 = 1; p_4 = p_5 = p_6 = 1$. In the last case we note that

$$\lim_{t \rightarrow +\infty} e^{tadu_1} \wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_5 \wedge e_6 \wedge e_7 = \wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge X_{(1,9)} \wedge X_{(1,7)} \wedge X_{(1,5)}.$$

We want the last factor to be $X_{(1,4)} \wedge X_{(1,5)} \wedge X_{(1,6)}$ so apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} ad(X_{(4,-9)} + X_{(6,-7)})}$ and get the desired element. For the case of $p_4 = 2, p_5 = 1$. We apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(1,-8)}}$ to $\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_5 \wedge e_6 \wedge e_7$ and get

$$\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_6 \wedge e_7 \wedge X_{(1,4)}.$$

We next apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(1,-6)}}$ and get $\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_7 \wedge X_{(1,4)} \wedge X_{(2,4)}$ we note that we have a “carry”.

$$\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_7 \wedge X_{(1,4)} \wedge X_{(2,4)} = [\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge X_{(1,5)} \wedge X_{(1,4)} \wedge X_{(2,4)}].$$

There is one more case $p_4 = 3$. Here we start with $\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge e_6 \wedge e_7 \wedge X_{(1,4)}$. Now we must avoid the carry so apply $\lim_{t \rightarrow +\infty} e^{tadu_4}$ and get

$$\wedge^{(4)}_{(2)} \mathfrak{a}^{(3)} \wedge X_{(1,4)} \wedge X_{(2,4)} \wedge X_{(4,4)}.$$

Now apply $\lim_{t \rightarrow 0} e^{\frac{1}{t} \text{ad} X_{(3,-4)}}$ and get $\wedge^{\binom{4}{2}} \mathfrak{a}^{(3)} \wedge X_{(1,4)} \wedge X_{(2,4)} \wedge X_{(3,4)}$ as desired. We will leave it to the reader to set up the corresponding inductions, carries to prove the result in this case.

D_n , $n \geq 4$. As we shall see, this case is almost identical with that of B_n with one complication (involving the Pfaffian) that we will study now. Here we will use the realization in section 2. We write (as in the case of B_n)

$$X_{(i,-j)} = X_{\varepsilon_i - \varepsilon_j} = E_{ij} - E_{2n+1-j, 2n+1-i} \text{ for } 1 \leq i < j \leq n$$

and

$$X_{(i,j)} = X_{\varepsilon_i + \varepsilon_j} = E_{i, 2n+1-j} - E_{j, 2n+1-i} \text{ for } 1 \leq i < j \leq n.$$

If $i > j$ then $X_{(i,j)} = -X_{(j,i)}$. Then $e = \sum_{i=1}^{n-1} X_{\varepsilon_i - \varepsilon_{i+1}} + X_{\varepsilon_{n-1} + \varepsilon_n}$. With this notation we have

$$\nabla \text{pfaff}(e) = c_n (X_{\varepsilon_1 - \varepsilon_n} - X_{\varepsilon_1 + \varepsilon_n}).$$

Indeed, let ζ be an outer automorphism of $\mathfrak{g}$ such that $\zeta \varepsilon_i = \varepsilon_i, i < n$ and $\zeta \varepsilon_n = -\varepsilon_n$. Then we may assume that $\zeta e = e$. Under this condition we have observed that $\zeta \text{pfaff} = -\text{pfaff}$. We also note (if A is an $m \times m$ matrix that A_{ij} is its ij entry)

Lemma 15 $(e^{n-1})_{1n} = (e^{n-1})_{1n+1} = 1$.

Proof. We write

$$e = \begin{bmatrix} J & M \\ 0 & -J \end{bmatrix}$$

with J the standard $n \times n$ nilpotent Jordan block $\sum E_{i, i+1}$ and $M = E_{n-1, 1} - E_{n, 2}$. Then

$$e^k = \begin{bmatrix} J^k & M_k \\ 0 & (-1)^k J^k \end{bmatrix}$$

and $M_{k+1} = JM_k + (-1)^k MJ^k$. We note that $E_{i,j}J$ is $E_{i,j+1}$ or 0. Thus only the first term contains $E_{n-k,i}$ for some i . This implies that in the expansion of M_{n-1} indicated by the above recursion the only term that has a nonzero entry in the first row is $J^{n-1}M$. This implies that the first row of e^{n-1} is all 0's except for 1's in the n and $n+1$ position. This proves the lemma. ■

We use the same notation as in the case of B_n . We label the e_j so that they are in non-decreasing order by the eigenvalue of $\text{ad}(h)$ and if n is even the gradient of the Pfaffian will have the smaller index. We will also use the notation u_i for the element of $\mathfrak{h}$ such that $\varepsilon_j(u_i) = \delta_{ij}$.

If n is even $n = 2k$ then up to scalar multiple we have

$$\begin{aligned} e_1 \wedge e_2 \wedge \cdots \wedge e_n &= e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge X_{(1,-n)} \wedge X_{(1,n)} \wedge e_{k+2} \wedge \cdots \wedge e_n + \\ e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge X_{(1,-n)} \wedge z \wedge e_{k+2} \wedge \cdots \wedge e_n &- e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge X_{(1,n)} \wedge z \wedge e_{k+2} \wedge \cdots \wedge e_n. \end{aligned}$$

With $z = \sum_{i=2}^k X_{(i,i+1)}$. Using this we can write out the matrix of supports in this case. This time we set (here n can be even or odd).

$$\phi_n(r) = \begin{cases} -r, 1 \leq r \leq n \\ 2n+2-r, n+2 \leq r \leq 2n \end{cases}$$

we set $a_{i,j} = (i, \phi_n(2j+i-1))$ if $j \neq k, k+1, 2j+i-1 \neq n+1, a_{i,j} = (i-1, n)$ if $2j+i-1 = n+1$ and $i+j \leq n+1$. the matrix of supports with the k and $k+1$ column deleted is given by $A = (a_{ij})$ thus if $n = 10$ then

$$A = \begin{array}{cccccccc} (1, -2) & (1, -4) & (1, -6)) & (1, -8) & * & * & (1, 8) & (1, 6) & (1, 4) & (1, 2) \\ (2, -3) & (2, -5) & (2, -7) & (2, -9) & * & * & (2, 7) & (2, 5) & (2, 3) & \\ (3, -4) & (3, -6) & (3, -8) & (3, -10) & * & * & (3, 6) & (3, 4) & & \\ (4, -5) & (4, -7) & (4, -9) & (3, 10) & * & * & (4, 5) & & & \\ (5, -6) & (5, -8) & (5, -10) & (4, 9) & * & * & & & & \\ (6, -7) & (6, -9) & (5, 10) & (5, 8) & * & * & & & & \\ (7, -8) & (7, -10)) & (6, 9) & (6, 7) & & & & & & \\ (8, -9) & (7, 10) & (7, 8) & & & & & & & \\ (9, -10) & (8, 9) & & & & & & & & \\ (9, 10) & & & & & & & & & \end{array}.$$

The blank columns correspond to $X_{(1,-10)} \wedge X_{(1,10)} + X_{(1,-10)} \wedge z - X_{(1,10)} \wedge z$ with $z = X_{(2,9)} + X_{(3,8)} + X_{(4,7)} + X_{(5,6)}$.

If $n = 2k+1$ then we set for $i+j \leq n+1, a_{i,j} = (i, \phi_n(2j+i-1))$ if $j \neq k+1$ and $2j+i-1 \neq n+1$ if $2j+i-1 = n+1$ then $a_{i,j} = (i-1, n)$. If $j = k+1$ then the rules apply except all entries a_{ij} are blank except for $i = 1, 2$. Thus if $n = 9$ then the matrix of supports is given by

$$\begin{array}{cccccccc} (1, -2) & (1, -4) & (1, -6)) & (1, -8) & (1, -9) & (1, 8) & (1, 6) & (1, 4) & (1, 2) \\ (2, -3) & (2, -5) & (2, -7) & (2, -9) & (1, 9) & (2, 7) & (2, 5) & (2, 3) & \\ (3, -4) & (3, -6) & (3, -8) & (2, 9) & * & (3, 6) & (3, 4) & & \\ (4, -5) & (4, -7) & (4, -9) & (3, 8) & * & (4, 5) & & & \\ (5, -6) & (5, -8) & (4, 9) & (4, 7) & * & & & & \\ (6, -7) & (6, -9) & (5, 8) & (5, 6) & & & & & \\ (7, -8) & (6, 9)) & (6, 7) & & & & & & \\ (8, -9) & (7, 8) & & & & & & & \\ (8, 9) & & & & & & & & \end{array}$$

We now begin the proof of the main theorem for D_n . We will assume that the reader is familiar with the techniques used for types A,B and C. Let α correspond to

$$\{\varepsilon_1 - \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_n\}$$

we will first prove that it is in the closure of $B\mathfrak{g}^e$. There are two cases n even and n odd. We first look at the case when $n = 2k$. We have

$$\lim_{t \rightarrow +\infty} e^{tad(u_1)} [e_1 \wedge \dots \wedge e_n] = x = [X_{(1,-2)} \wedge X_{(1,-4)} \wedge \dots \wedge X_{(1,-n)} \wedge X_{(1,n)} \wedge X_{(1,n-2)} \wedge \dots \wedge X_{(1,2)}].$$

If we successively calculate

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(2,3)}} x = x_1, \lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(4,5)}} x_1 = x_2, \dots, \lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(n-2, n-1)}} x_{k-1} = x_k.$$

Then

$$x_k = \mathbf{a}.$$

If $n = 2k + 1$ then the first step yields

$$\lim_{t \rightarrow +\infty} e^{t ad(u_1)} [e_1 \wedge \dots \wedge e_n] = x$$

$$= [X_{(1,-2)} \wedge X_{(1,-4)} \wedge \dots \wedge X_{(1,-n+1)} \wedge (X_{(1,-n)} - X_{(1,n)}) \wedge X_{(1,n-1)} \wedge \dots \wedge X_{(1,2)}].$$

If we do the sequence

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(2,3)}} x = x_1, \lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(4,5)}} x_1 = x_2, \dots, \lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(n-1, n)}} x_k = x_{k+1}.$$

Then the last element is the desired one.

Since the outer automorphism $u_i \mapsto u_i, i < n$ and $u_n \mapsto -u_n$ preserves $\mathfrak{g}^e$ and B this implies that to prove the theorem in this case we need only consider the Λ of the form

$$\{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_{m_1}\} \cup \{\varepsilon_2 + \varepsilon_3, \dots, \varepsilon_2 + \varepsilon_{m_2}\} \cup \dots \cup \{\varepsilon_r + \varepsilon_{r+1}, \dots, \varepsilon_r + \varepsilon_{m_r}\}.$$

with $m_1 \geq \dots \geq m_r, m_i > i$ and the sum $\sum_{i=1}^r (m_i - i) = n$. Exactly as in the case of B_n .

We will do the case $m_1 = n$ separately. In this case $\Lambda = \{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_n\} \cup \{\varepsilon_2 + \varepsilon_3\}$. We first note that since $n \geq 4$

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad(X_{(1,4)})} [e_1 \wedge \dots \wedge e_n] = [e_2 \wedge \dots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)}].$$

If $n = 4$ then $e_2 \wedge e_3$ is up to scalar multiple $X_{(1,-4)} \wedge X_{(1,4)} + c(X_{(1,-4)} - X_{(1,4)}) \wedge X_{(2,3)}$. Thus

$$\lim_{t \rightarrow +\infty} e^{t ad u_4} [e_2 \wedge e_3 \wedge X_{(1,3)} \wedge X_{(1,2)}] = [X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}]$$

as desired.

If $n \geq 5$ then $e^{\frac{1}{t} adX_{35}} e_2 = e_2 + c \frac{1}{t} X_{(2,3)}$ and $e^{\frac{1}{t} adX_{35}} e_3 \wedge \dots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)} = e_3 \wedge \dots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)}$. Thus

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{35}} [e_2 \wedge \dots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)}] = [e_3 \wedge \dots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}].$$

We look at the special case $n = 5$. We have (up non-zero scalar) $e_3 = X_{(1,-5)} - X_{(1,5)}$, and

$$[e_4 \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] = [X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}].$$

Thus as above

$$\lim_{t \rightarrow +\infty} e^{tadu_5} [e_3 \wedge e_4 \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] = [X_{(1,5)} \wedge X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}].$$

We now consider $n = 2k$, $k > 2$. Then we have

$$\begin{aligned} w &= \lim_{t \rightarrow +\infty} e^{tadu_1} [e_3 \wedge \cdots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] = \\ &[X_{(1,-6)} \wedge \cdots \wedge X_{(1,-n+2)} \wedge (X_{(1,-n)} \wedge X_{(1,n)} + (X_{(1,-n)} - X_{(1,n)}) \wedge z) \wedge X_{(1,n-2)} \wedge \cdots \wedge X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] \\ &\text{with } \sup z = \{\varepsilon_2 + \varepsilon_{n-1}, \dots, \varepsilon_k + \varepsilon_{k+1}\}. \text{ Now take } \lim_{t \rightarrow +\infty} e^{tadu_n} w \text{ and get} \\ &[X_{(1,-6)} \wedge \cdots \wedge X_{(1,-n+2)} \wedge X_{(1,-n)} \wedge X_{(1,n)} \wedge X_{(1,n-2)} \wedge \cdots \wedge X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}]. \end{aligned}$$

If we now argue in the first case above we have the desired outcome.

If $n = 2k + 1$ with $k \geq 3$ then

$$\begin{aligned} w &= \lim_{t \rightarrow +\infty} e^{tadu_1} [e_3 \wedge \cdots \wedge e_{n-1} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] = \\ &[X_{(1,-6)} \wedge \cdots \wedge X_{(1,-n+1)} \wedge (X_{(1,-n)} - X_{(1,n)}) \wedge X_{(1,n-1)} \wedge \cdots \wedge X_{(1,4)} \wedge X_{(1,3)} \wedge X_{(1,2)} \wedge X_{(2,3)}] \end{aligned}$$

As above we calculate $\lim_{t \rightarrow +\infty} e^{tadu_n} w$ and use the argument at the beginning of this discussion to “fill in” the elements $X_{i,2j+1}$ with $j = 2, \dots, k-1$.

We may now assume that Λ is given as above and $m_1 < n$. We note that $m_1 > r$.

Before we continue we will use the same description of the sets $\Lambda \subset \Phi^+$ that satisfy condition 2. in section 3 that are still in question as we did in the case of B_n . We note that the set must contain $\Lambda^{(r)} = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq r \text{ and } i+1 \leq j \leq r+1\}$. The rest of Λ is given as $\cup_{j=1}^{m_1-r-1} \{\varepsilon_i + \varepsilon_{j+r+1} \mid 1 \leq i \leq p_j\}$ and

$$r \geq p_1 \geq p_2 \geq \cdots \geq p_{m_1-r-1} \geq 0$$

satisfies $\sum_j p_j = |\Lambda| - \frac{r(r+1)}{2}$. Our strategy in the proof is, as in the case of B_n , to use this decomposition of the set and handle the two parts individually.

Let $n = 2k$ or $2k + 1$ and set $\nu = n - 2k$. We prove by induction on r that if $\mathfrak{a}^{(r)}$ is the ideal corresponding to $\Lambda^{(r)}$ then if $r+1$ is even and if the inequality $\binom{r}{2} + 1 - \frac{r-1}{2} \leq k + \nu \leq \binom{r}{2} + \frac{r+1}{2}$ is not satisfied then

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge e_{\binom{r+1}{2}+1-\frac{r+1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}$$

is in the closure of $B\mathfrak{g}^e$. If the inequality is satisfied then

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge X_{(1,r+2)} \wedge e_{\binom{r+1}{2}+2-\frac{r+1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}$$

is in the closure of $B\mathfrak{g}^e$. If $r+1$ is odd and $n > \binom{r+1}{2}$ then

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge X_{(1,r+2)} \wedge e_{\binom{r+1}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{n-\frac{r+2}{2}}$$

is in the closure of $B\mathfrak{g}^e$ or if $n = \binom{r+1}{2}$. then $\mathfrak{a}^{(r)}$ is in the closure.

If $r = 1$ then $\mathfrak{a}^{(r)} = \mathbb{C}X_{12} = \mathbb{C}e_n$, $\binom{r+1}{2} + 1 - \frac{r+1}{2} = 1$ and $n - \frac{r+1}{2} = n - 1$. Thus the assertion in this case is $\mathfrak{g}^e$ is in the closure of $\mathfrak{g}^e$. If $r = 2$ then $\Lambda^{(2)} = \{\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$. We are assuming that $n > 3$. As above we have

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{14}} [e_1 \wedge \cdots \wedge e_n] = [X_{(1,2)} \wedge X_{(1,3)} \wedge e_2 \wedge \cdots \wedge e_{n-1}].$$

If $n = 4$ then as above up to scalar multiple

$$e_2 \wedge e_3 = X_{(1,-4)} \wedge X_{(1,4)} + aX_{(1,-4)} \wedge X_{(2,3)} - aX_{(1,4)} \wedge X_{(2,3)}$$

with $a \neq 0$ thus

$$e^{tadu_4} e_2 \wedge e_3 = X_{(1,-4)} \wedge X_{(1,4)} + aX_{(1,-4)} \wedge X_{(2,3)} - aX_{(1,4)} \wedge X_{(2,3)}$$

hence

$$\lim_{t \rightarrow \infty} e^{tadu_4} [X_{(1,2)} \wedge X_{(1,3)} \wedge e_2 \wedge \cdots \wedge e_{n-1}] = [X_{(1,2)} \wedge X_{(1,3)} \wedge X_{(1,4)} \wedge X_{(2,3)}]$$

as asserted for this case. We also note that this completes the proof for the special case of $n = 4$. If $n > 4$ then one checks that

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{24}} [X_{(1,2)} \wedge X_{(1,3)} \wedge X_{(2,3)} \wedge X_{(1,4)} \wedge e_3 \wedge \cdots \wedge e_{n-2}]$$

as asserted. We now begin the inductive step. So we assume the result for $r - 1 \geq 3$ and we prove it for r . We must assume that $n \geq \binom{r+1}{2}$. First we assume that $r + 1$ is even. Then r is odd and the inductive hypothesis implies that

$$\wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge e_{\binom{r}{2}+1-\frac{r-1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}$$

is in the closure of $B\mathfrak{g}^e$. If $n = 2k$ or $2k + 1$ and $\binom{r}{2} + 1 - \frac{r-1}{2} > k + 1$ then the argument is identical with that in the case of B_n . If $\binom{r}{2} + 1 - \frac{r-1}{2} < k - r + 1$ if n is even or if $\binom{r}{2} + 1 - \frac{r-1}{2} < k + 2 - r$ if n is odd then the argument is essentially the same as the previous two cases. Set $\nu = n - 2k$. The new aspect of the argument is if

$$\binom{r}{2} + 1 - \frac{r-1}{2} \leq k + \nu \leq \binom{r}{2} + \frac{r+1}{2}$$

since then the method in the case of B_n involves the gradient of the Pfaffian. We note that under these circumstances these inequalities are inconsistent with $n = \binom{r+1}{2}$ and $n \geq 6$ (we have proved the theorem for $n \leq 5$). Thus $n > \binom{r+1}{2}$. We follow the method until we come to

$$u = \wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge \cdots \wedge X_{(s,r+1)} \wedge e_{k+\nu} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}.$$

Now

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} ad X_{(r+2,n)}} u = \wedge \binom{r}{2} \mathfrak{a}^{(r-1)} \wedge X_{(1,r+1)} \wedge \cdots \wedge X_{(s,r+1)} \wedge X_{(1,r+2)} \wedge e_{k+\nu+1} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}.$$

We now continue as before and have

$$\wedge^{\binom{r+1}{2}} \mathfrak{a}^{(r)} \wedge X_{(1,r+2)} \wedge e_{\binom{r+1}{2}+2-\frac{r+1}{2}} \wedge \cdots \wedge e_{n-\frac{r+1}{2}}.$$

For the assertion for $r+1$ odd then if on the previous step we had to use the argument above then we can do the inductive step as in the case of B_n without the first limit. If the inequality

$$\binom{r}{2} + 1 - \frac{r}{2} \leq k + \nu \leq \binom{r}{2} + \frac{r+2}{2}$$

is satisfied then the inductive hypothesis implies that

$$u = \wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge e_{\binom{r}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{n-\frac{r}{2}}$$

is in the closure of $B\mathfrak{g}^e$. We first do

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX_{(r+2,n)}} u$$

which yields

$$\wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(2,r+1)} \wedge X_{(1,r+2)} \wedge e_{\binom{r}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_{k+\nu-1} \wedge (X_{(1,n)} + z) \wedge \cdots \wedge e_{n-\frac{r}{2}}$$

(z is at the beginning of this interminable case) if n is even and

$$\wedge^{\binom{r}{2}} \mathfrak{a}^{(r-1)} \wedge X_{(2,r+1)} \wedge X_{(1,r+2)} \wedge e_{\binom{r}{2}+1-\frac{r}{2}} \wedge \cdots \wedge e_k \wedge e_{k+2} \wedge \cdots \wedge e_{n-\frac{r}{2}}$$

if n is odd. Now continue as in the case of B_n .

The hearty reader is encouraged to use the above tables for $n = 9$ and 10 to carry out all the techniques in these two cases.

E₈: We must show that the two 8 dimensional ideals in $\mathfrak{b}$ are contained in the closure of $B\mathfrak{g}^e$. We will use similar methods for the two examples. Let V be as in the **E₈** subsection of section 3. We will choose a non-zero root vector for each positive root. We denote this element by $X(\alpha)$ if α is the root and if say $\alpha = (1, 1, 1, 1, 0, 0, 0, 0)$ then we write $X(1, 1, 1, 1, 0, 0, 0, 0)$. We write $e_1, \dots, e_8$ with $e_1 = e$ for a basis of $\mathfrak{g}^e$ with $[H, e_i] = m_i e_i$ with $(m_1, \dots, m_8) = (1, 7, 11, 13, 17, 19, 23, 29)$. Consider the subset

$$\begin{aligned} Z = & \{(2, 3, 4, 5, 4, 3, 2, 1), (2, 3, 4, 6, 4, 3, 2, 1)\} \cup \\ & \{(2, 3, 4, 6, 5, 3, 2, 1), (2, 3, 4, 6, 5, 4, 2, 1)\} \cup \\ & \{(2, 3, 4, 6, 5, 4, 3, 1), (2, 3, 4, 6, 5, 4, 3, 2)\} \end{aligned}$$

of V . Let $\mathfrak{a}$ be the ideal in $\mathfrak{b}$ that is the sum of these roots spaces. We set $\beta_1 = (2, 3, 4, 6, 5, 4, 2, 1), \beta_2 = (1, 2, 3, 4, 4, 3, 2, 1), \beta_3 = (1, 1, 2, 3, 3, 2, 2, 1), \beta_4 = (1, 1, 2, 3, 2, 1, 1, 1), \beta_5 = (1, 1, 2, 2, 1, 0, 0, 0)$. Then

$$\lim_{t_5 \rightarrow 0} e^{\frac{1}{t_5} adX(\beta_5)} \lim_{t_4 \rightarrow 0} e^{\frac{1}{t_4} adX(\beta_4)} \lim_{t_3 \rightarrow 0} e^{\frac{1}{t_3} adX(\beta_3)} \lim_{t_2 \rightarrow 0} e^{\frac{1}{t_2} adX(\beta_2)} \lim_{t_1 \rightarrow 0} e^{\frac{1}{t_1} adX(\beta_1)}$$

applied to $[e_1 \wedge \cdots \wedge e_8]$ yields

$$\wedge^6 \mathfrak{a} \wedge e_6 \wedge e_7.$$

Now (taking into account a “carry”)

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX(1,1,1,1,0,0,0,0)} \wedge^6 \mathfrak{a} \wedge e_6 \wedge e_7$$

is $\wedge^8$ of the ideal corresponding to $V \cup \{(2, 2, 4, 5, 4, 3, 2, 1)\}$. For the other ideal we have

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX(0,1,1,1,0,0,0,0)} \wedge^6 \mathfrak{a} \wedge e_6 \wedge e_7 = \wedge^6 \mathfrak{a} \wedge X(1, 3, 3, 5, 4, 3, 2, 1) \wedge e_7.$$

Now

$$\lim_{t \rightarrow +\infty} e^{tadH_2} \wedge^6 \mathfrak{a} \wedge X(1, 3, 3, 5, 4, 3, 2, 1) \wedge e_7$$

yields $\wedge^8$ of the ideal corresponding to $\{(1, 3, 3, 5, 4, 3, 2, 1)\} \cup V$ and completes the proof of the main result for E_8 .

E_7 : As in the case of E_8 we set

$$Z = \{(1, 2, 2, 3, 3, 2, 1), (1, 2, 2, 4, 3, 2, 1)\} \cup \{(1, 2, 3, 4, 3, 2, 1), (2, 2, 3, 4, 3, 2, 1)\}$$

and we set $\mathfrak{a}$ equal to the ideal corresponding to Z . We define $H_i \in \mathfrak{h}$ by $\alpha_j(H_i) = \delta_{ij}$. Set $\beta_1 = (1, 2, 2, 4, 3, 2, 1), \beta_2 = (1, 1, 2, 3, 2, 1, 0), \beta_3 = (1, 1, 1, 1, 1, 1, 1)$

$$\lim_{t_3 \rightarrow 0} e^{\frac{1}{t_3} adX(\beta_3)} \lim_{t_2 \rightarrow 0} e^{\frac{1}{t_2} adX(\beta_2)} \lim_{t_1 \rightarrow 0} e^{\frac{1}{t_1} adX(\beta_1)} [e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7] =$$

$$\wedge^4 \mathfrak{a} \wedge e_4 \wedge e_5 \wedge e_6.$$

In order to see the ideal, $\mathfrak{u}$, corresponding to

$$\Lambda_1 = \{(1, 2, 2, 3, 2, 1, 0), (1, 2, 2, 3, 2, 1, 1), (1, 2, 2, 3, 2, 2, 1)\} \cup Z$$

in the closure of $B\mathfrak{g}^e$. We set $\gamma_1 = (1, 1, 2, 2, 1, 1, 1), \gamma_2 = (1, 2, 2, 3, 2, 1, 0)$ and $\gamma_3 = (1, 2, 2, 3, 2, 2, 1)$

$$\lim_{t \rightarrow +\infty} e^{tad(H_1+H_2+H_3)} \wedge^4 \mathfrak{a} \wedge e_4 \wedge e_5 \wedge e_6 = \wedge^4 \mathfrak{a} \wedge X(\gamma_1) \wedge X(\gamma_2) \wedge X(\gamma_3)$$

finally

$$\lim_{t \rightarrow 0} e^{\frac{1}{t} adX(0,1,0,1,0,0,0)} \wedge^4 \mathfrak{a} \wedge X(\gamma_1) \wedge X(\gamma_2) \wedge X(\gamma_3) = \wedge^7 \mathfrak{u}.$$

The other two cases are

$$\Lambda_2 = \{(1, 1, 2, 3, 2, 2, 1), (1, 1, 2, 3, 3, 2, 1), (1, 2, 2, 3, 2, 2, 1)\} \cup Z$$

and

$$\Lambda_3 = \{(1, 2, 2, 3, 2, 1, 1), (1, 1, 2, 3, 3, 2, 1), (1, 2, 2, 3, 2, 2, 1)\} \cup Z.$$

We have (since there is a “carry”)

$$\lim_{t \rightarrow 0} e^{\frac{1}{t}X(1,0,1,1,1,0,0)} \wedge^4 \mathfrak{a} \wedge e_4 \wedge e_5 \wedge e_6 =$$

$$\wedge^6 \mathfrak{v} \wedge e_5$$

with $\mathfrak{v}$ the ideal corresponding to $\{(1, 1, 2, 3, 3, 2, 1), (1, 2, 2, 3, 2, 2, 1)\} \cup Z$. If $\mathfrak{u}_i$ corresponds to Λ_i for $i = 2, 3$ then

$$\lim_{t \rightarrow 0} e^{\frac{1}{t}adX(0,0,0,1,0,0,0)} \wedge^6 \mathfrak{v} \wedge e_5 = \wedge^7 \mathfrak{u}_2$$

and

$$\lim_{t \rightarrow 0} e^{\frac{1}{t}adX(0,1,0,0,0,0,0)} \wedge^6 \mathfrak{v} \wedge e_5 = \wedge^7 \mathfrak{u}_3.$$

This completes the proof for E_7 .

E_6 : In this case we set

$$V = \{(1, 1, 1, 2, 2, 1), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 2, 1)\} \cup$$

$$\{(1, 1, 2, 3, 2, 1), (1, 2, 2, 3, 2, 1)\}$$

Then if $\gamma_1 = (0, 1, 1, 2, 2, 1), \gamma_2 = (1, 1, 2, 2, 1, 0), \gamma_3 = (1, 1, 1, 2, 1, 1)$ then the three ideals in question are $\Lambda_i = \{\gamma_i\} \cup V$ for $i = 1, 2, 3$. Let $\mathfrak{u}$ be the ideal in $\mathfrak{h}$ corresponding to V . Set $\beta_1 = (1, 1, 2, 3, 2, 1), \beta_2 = (1, 1, 1, 1, 1, 0), \beta_3 = (0, 0, 0, 1, 1, 1)$ then

$$\lim_{t_3 \rightarrow 0} e^{\frac{1}{t_3}adX(\beta_3)} \lim_{t_2 \rightarrow 0} e^{\frac{1}{t_2}adX(\beta_2)} \lim_{t_1 \rightarrow 0} e^{\frac{1}{t_1}adX(\beta_1)} [e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6] = \wedge^5 \mathfrak{u} \wedge e_4$$

we have shown that the support of e_4 is $\{\gamma_1, \gamma_2, \gamma_3\}$. Let $\mathfrak{a}_i$ be the ideal corresponding to Λ_i . Let $H_i \in \mathfrak{h}$ be defined by $\alpha_j(H_i) = \delta_{ij}$. Set $v_1 = H_2 + H_3 + H_4 + H_5 + H_6, v_2 = H_1 + H_2 + H_3 + H_4 + H_5$ and let $v_3 = H_1 + H_6$. Then

$$\lim_{t \rightarrow +\infty} e^{tadv_i} \wedge^5 \mathfrak{u} \wedge e_4 = \wedge^6 \mathfrak{a}_i.$$

This completes the proof of Theorem 14.

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