Let $G$ be a commutative group. We define $\hat{G}$ to be the set of all group homomorphisms from $G$ as a group under addition to $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Such homomorphisms are called characters. We have defined a group structure on $\hat{G}$ as follows: If $\chi_1, \chi_2 \in \hat{G}$ then

$$(\chi_1 \chi_2)(x) = \chi_1(x)\chi_2(x).$$

The identity in the group is 1 which is our notation for the function $\chi(x) = 1$ for all $x \in G$.

If $G = R_m = \mathbb{Z}/m\mathbb{Z}$, for $m > 0$ then every character of $R_m$ is of the form

$$\chi_j(x) = e^{\frac{2\pi i j x}{m}}.$$

With $j = 0, 1, \ldots, m-1$ and $x$ is taken as an element of $\{0, 1, \ldots, m-1\}$. We note that $\chi_j$ depends only on its equivalence class modulo $m$.

If $y \in R_m$ define a function $\eta_y : \hat{R}_m \rightarrow \mathbb{C}^\times$ by $\eta_y(\chi) = \chi(y)$.

1) Prove that $\eta_y$ defines a character of $\hat{R}_m$.

2) Prove that if $\eta$ is a character of $\hat{R}_m$ then there exists $y \in R_m$ such that $\eta = \eta_y$.

3) Prove that the mapping $\varphi : R_m \rightarrow \hat{R}_m$ is a group isomorphism (here $R_m$ is considered to be a group under addition).

4) Prove that if $\chi \in \hat{R}_m$ then

$$\sum_{x \in R_m} \chi(x) = \begin{cases} m & \text{if } \chi = 1 \\ 0 & \text{if } \chi \neq 1 \end{cases}.$$ (Hint: We may assume that $\chi = \chi_j$ for some $j$.)

5) Prove that if $x \in R_m$ then

$$\sum_{\chi \in \hat{R}_m} \chi(x) = \begin{cases} m & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$ 

If $f : R_m \rightarrow \mathbb{C}$ define $\mathcal{F}(f) : \hat{R}_m \rightarrow \mathbb{C}$ by

$$\mathcal{F}(f)(\chi) = \frac{1}{\sqrt{m}} \sum_{y \in R_m} \chi(y)^{-1} f(y).$$
6) Prove that
\[ f(x) = \frac{1}{\sqrt{m}} \sum_{\chi \in R_m} \chi(x) \mathcal{F}(f)(\chi) \]
for all \( f : R_m \to \mathbb{C} \) and all \( x \in R_m \). (Hint: Write out the right hand sum as a double sum and show that the double sum can be written
\[
\frac{1}{m} \sum_{\chi \in R_m} \sum_{y \in R_m} \chi(x - y) f(y).
\]
Make the change of variables \( z = x - y \) so \( y = x - z \) and write the sum
\[
\frac{1}{m} \sum_{z \in R_m} (\sum_{\chi \in R_m} \chi(z)) f(x - z).
\]
Now use the results above.)

If \( f : R_m \to \mathbb{C} \) then we can define a new function \( \widehat{f}(j) = \mathcal{F}f(\chi_j) \) mapping \( R_m \) to \( \mathbb{C} \).

7) Write out the formula for \( \widehat{f}(j) \).

8) Prove that \( \widehat{f}(x) = f(-x) \).