1. Introduction

The theory of Whittaker vectors and Jacquet integrals was initiated in the thesis of Jacquet [J] (this is the reason for the name of the integrals) under the direction of Godement. The importance of this theory derives from the fact that the generic Fourier coefficients (suitably interpreted) of an automorphic form at a cusp can be expressed in terms of Whittaker vectors on the spaces of smooth vectors of admissible representations. Since a significant part of the application of the theory of automorphic forms to number theory involves these Fourier coefficients the theory of Whittaker vectors has a major role in the theory.

In this paper we give a definitive treatment of generic characters of unipotent radicals of a class of parabolic subgroups (called nice and defined in the next section) and prove a holomorphic continuation of Jacquet integrals for the induced representations from finite dimensional representations of the parabolic. In addition, an analogue of the multiplicity one theorem is proved. The precise forms of these theorems can be found in the next section with a discussion of earlier related work. We note that if the parabolic is not nice then our method gives an upper bound on the dimension of the space of “generic” Whittaker vectors defined on the space of $C^\infty$ vectors of a representation induced from a finite dimensional representation of it. We also observe that the class of nice parabolics are precisely those for which the vanishing theorem of Lynch holds (see Theorem 4).

The author is gratified that his earlier work in this subject [Wa1] is still being applied. In that paper the conclusion 1,2,3. in the next section was proved under very complicated hypotheses. Also, there was a conjecture which implied in essence the Corollary in the next section. This conjecture is false and the first counterexample appears in [Wa3] for split $G_2$ and thereby for almost all cases of a parabolic with a unipotent radical isomorphic with a Heisenberg group. Since that was exactly the case of interest in [Wa3] a new approach to the
2. Notation and main results.

Let \( G \) be a real reductive group of inner type with compact center and let \( P \) be a parabolic subgroup of \( G \). Let \( N \) be the unipotent radical of \( P \) and let \( K \) be a maximal compact subgroup of \( G \) with corresponding Cartan involution \( \theta \). Let \( P = MAN \) be a Langlands decomposition of \( P \). Let \( P_o = M_o A_o N_o \) be a minimal parabolic subgroup of \( G \) such that \( M_o \subset M, A \subset A_o, N \subset N_o \). Set \( \mathfrak{N} = \theta(\mathfrak{N}) \). We will use lower case fraktur letters for Lie algebras so \( \mathfrak{a} = \text{Lie}(A), \mathfrak{n} = \text{Lie}(N) \).

Let \((\sigma, H_\sigma)\) be a smooth Fréchet representation of \( M \) of moderate growth (for the purposes of this paper one can assume that \( H_\sigma \) is continuous and finite dimensional since then it is automatically a smooth Fréchet representation of moderate growth) and let \( \nu \in \mathfrak{a}_c^* \) then we can form the smooth induced representation

\[
I_{P,\sigma,\nu}^\infty = \{ f \in C^\infty(G; H_\sigma) \mid f(namg) = a^{\nu + \rho} \sigma(m) f(g), \] 
\[ n \in N, a \in A, m \in M, g \in G \}
\]

with the \( C^\infty \) topology and the action of \( G \) by right translation

\[
\pi_{P,\sigma,\nu}(g)f(x) = f(xg).
\]

We note that since \( G = PK \) if \( f \in I_{P,\sigma,\nu}^\infty \) then \( f|_K \in I_{\sigma|_{M \cap K}}^\infty \) (the smooth induced representation of \( \sigma|_{K \cap M} \) from \( K \cap M \) to \( K \)) and we have an inverse map. That is, if \( f \in I_{\sigma|_{M \cap K}}^\infty \) then we set \( f_{P,\sigma,\nu}(namk) = a^{\nu + \rho} \sigma(m)f(k), n \in N, a \in A, m \in M, k \in K \). Finally, let \( \chi : \mathfrak{N} \to S^1 \) be a unitary character of \( \mathfrak{N} \). If \( f \in I_{\sigma|_{M \cap K}}^\infty \) then we define (after having chosen a Haar measure on \( \mathfrak{N} \))

\[
J_\chi^{\mathfrak{N}} f = \int_{\mathfrak{N}} f_{P,\sigma,\nu}(n)\chi(n)^{-1}d\mathfrak{n}
\]

Fix \( B \), a \( G \)-invariant real valued bilinear form on \( \mathfrak{g} \) such that \( B(\theta X, X) < 0 \) for all \( X \in \mathfrak{g} \). Then \( B|_{\mathfrak{a}} \) is positive definite. We write \((...,...)\) for...
the dual form on $\mathfrak{a}^*$ and for the complex bilinear extension to the complexification. It is not hard to show that there exists a constant $C_\sigma$ (depending on $\sigma$) such that in the set of all $\nu$ satisfying
\[ \text{Re}(\nu, \alpha) > C_\sigma, \alpha \in \Phi(P, A) \]
the integral defining $J_{P, \sigma, \nu}(f)$ converges absolutely and uniformly on compacta. Thus defining for each $\nu$ satisfying the above inequality a continuous functional on $I_{\mathfrak{a}^*}^{\mathbb{C} \cap K}$.

One desires a meromorphic continuation of these integrals to all of $\mathfrak{a}_c^*$. To be precise here is our “wish list” for “generic characters” $\chi$:

1. The map $\nu \mapsto J_{P, \sigma, \nu}(f)$ extends to a holomorphic map of $\mathfrak{a}_c^*$ to $H_\sigma$ for all $f \in I_{\mathfrak{a}^*}^{\mathbb{C} \cap K}$.

2. Set $Wh_\chi(I_{\mathfrak{a}^*}^{\mathbb{C} \cap K}) = \{ T \in (I_{\mathfrak{a}^*}^{\mathbb{C} \cap K})' | T(\pi_{P, \sigma, \nu}(\pi))f = \chi(\pi)^{-1}T(f), \pi \in \mathbb{N}, f \in I_{\mathfrak{a}^*}^{\mathbb{C} \cap K} \}$. Here the upper prime means continuous functionals. Then if $T \in Wh_\chi(I_{\mathfrak{a}^*}^{\mathbb{C} \cap K})$ then there exists $\lambda \in (H_\sigma)'$ such that $T = \lambda \circ J_{P, \sigma, \nu}$.

3. Set $M_\chi = \{ m \in M | \chi(m \pi m^{-1}) = \chi(\pi) \text{ for all } \pi \in \mathbb{N} \}$. Then we can define an action of $M_\chi$ on $Wh_\chi(I_{\mathfrak{a}^*}^{\mathbb{C} \cap K})$ via $\pi_{P, \sigma, \nu}(m)T = T \circ \pi_{P, \sigma, \nu}(m^{-1})$. The map $(H_\sigma)' \rightarrow Wh_\chi(I_{\mathfrak{a}^*}^{\mathbb{C} \cap K})$ given by $\lambda \mapsto \lambda \circ J_{P, \sigma, \nu}$ is an equivalence of representations of $M_\chi$.

Let $\chi : \mathbb{N} \rightarrow S^1$ be a unitary character. Then there exists $x \in \mathfrak{n}(= \text{Lie}(N))$ such that $d\chi(Y) = iB(x, Y)$ for $Y \in \theta \mathfrak{n}$. The only reasonable notion of generic is

**Definition 1.** A unitary character, $\chi$, of $\mathbb{N}$ is said to be non-degenerate if $d\chi(Y) = iB(x, Y)$ for $Y \in \theta \mathfrak{n}$ and $\text{Ad}(P)x$ is open in $\mathfrak{n}$.

In other words the orbit of $x$ is a Richardson element in $\mathfrak{n}_C$. We note that there exist parabolic groups such that there are no such non-degenerate characters of $\mathbb{N}$. Here is an example due to McGovern. Let $G = SL(5, \mathbb{R})$ and let $P$ be the parabolic subgroup consisting of matrices of the form
\[
\begin{bmatrix}
* & * & * & * \\
* & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & *
\end{bmatrix}
\]
one can show by a direct calculation that of \( x \in \mathfrak{n} \) and if \( B(x, \theta[\mathfrak{n}, \mathfrak{n}]) = 0 \) then \( [\mathfrak{p}, x] \neq \mathfrak{n} \). We will say that a parabolic subgroup of \( G \) is “nice” if there exists \( x \in \mathfrak{n} \) such that \( B(x, \theta[\mathfrak{n}, \mathfrak{n}]) = 0 \) then \( [\mathfrak{p}, x] = \mathfrak{n} \). If \( P \) is nice than our condition of nondegenerate is generic (i.e. defines an open dense subset of the characters). We also note that this notion of non-degenerate is the same as that in [Wa1] (see Lemma 6 below) so all of the examples in section 4 of [Wa1] are examples of nice parabolic subgroups.

The main result of this paper is

**Theorem 2.** If \( \chi \) is a non-degenerate character of \( \mathcal{N} \) and if \( \dim H_\sigma < \infty \) then 1.,2.,3. above are true.

Checking the non-degenerate condition is not necessarily easy to verify. We will now describe a corollary to the main theorem that is more applicable and is the essence of a conjecture in [Wa1] (more will be said about after the statement). We assume that \( y \in \theta \mathfrak{n} \) has the property that \( B(y, [\mathfrak{n}, \mathfrak{n}]) = 0 \). We define \( \chi(\exp X) = e^{iB(y,X)} \) for \( X \in \mathfrak{n} \). We assume that there exists \( x \in \mathfrak{n} \) such that \( [x, y] = h \in \mathfrak{n}, [h, x] = 2x, [h, y] = -2y \) and that the space \( \{ X \in \mathfrak{n} | [h, X] = 2X \} \) generates \( \mathfrak{n} \) as a Lie algebra. Then, if we choose \( k \in (\exp(\pi(x - y)/2)P) \cap K \), we have

\[
kPk^{-1} = \theta(P).
\]

We write

\[
J_{\sigma,\nu}^\chi(f) = \int_N f_{P,\sigma,\nu}(k^{-1}n)\chi(n)^{-1}dn.
\]

Notice that we have apparently used the same notation for two different objects however this is not so since that type of characters are different. Hopefully, there will be no confusion. We have

**Corollary 3.** Assuming the discussion above and replacing \( \mathcal{N} \) by \( N \) if \( \dim H_\sigma < \infty \) then 1.,2. and 3. are true.

This result has been proved in many special cases and our notion of generic agrees with all of the previous definitions. If \( P \) is minimal and \( G \) split over \( \mathbb{R} \), Jacquet[J] has proved 1.,2.,3. and for complex groups he has proved 1. for \( f \) a \( K \)-finite function and the essence of 3. which his multiplicity one theorem. If \( G \) is of rank one over \( \mathbb{R} \) this was proved by Schiffman[S]. For general real groups and minimal parabolic subgroup there are papers [Ha1,Ha2] that do special cases, there is the announcement of Varadarajan for Harish-Chandra 1983 and there is also the independent work of the author which appears in [Wa1]. This
paper goes beyond the case of minimal parabolic subgroup. The basic ideas in that paper will be used to prove the main theorem. \cite{Wa1} contained many examples of the main theorem including most cases when the nilpotent radical is abelian. The proofs in that paper used as hypotheses a set of conditions on certain double cosets in $G$ and then involved standard Bruhat theory a vanishing theorem for Lie algebra cohomology essentially due to Lynch, and an explicit method of construction of Whittaker vectors in tensor products with finite dimensional representations. In \cite{Wa3} this result was proved in the special case of the Heisenberg parabolic for the quaternionic real form of type $A, B, D, E, F, G$. Here the Bruhat theoretic method in \cite{Wa1} does not directly apply to these cases. It is here that the result of [K-V] came to the rescue. The above corollary applies directly to all of the examples in \cite{Wa1} section 4.

3. The category $\mathcal{W}_\psi$.

Let $G$ and $P = MAN$ be as in the introduction (we will maintain all of the notation therein). Let $x \in \mathfrak{n}$ be such that $B(x, [\theta \mathfrak{n}, \theta \mathfrak{n}]) = 0$. Then we can define a Lie algebra homomorphism of $\theta \mathfrak{n}$ to $i \mathbb{R}$ by $\psi(Y) = iB(x,Y)$. Let $\mathcal{W}_\psi$ denote the category of all $\mathfrak{g}$-modules, $M$, such that if $m \in M$, $Y \in \theta \mathfrak{n}$ there exits $k$ such that $(Y - \psi(Y))^k m = 0$. The following is a reformulation of Theorem 2.2 of \cite{Wa1}.

**Theorem 4.** Assume that $Ad(P)x$ is open in $\mathfrak{n}$. Then $H^i(\theta \mathfrak{n}, M \otimes \mathbb{C}_{-\psi}) = 0$ for all $i > 0$.

To prove this we need only show that $x$ satisfies the conditions (1),(2) in the beginning of section 2 in \cite{Wa1}. (Here we note that the roles of $\mathfrak{n}$ and $\theta \mathfrak{n}$ have been reversed). We must therefore show that

1. There is an element $h \in \mathfrak{a}$ such that $ad(h)$ has eigenvalues included in $\{0, \pm 2, \pm 4, ...\}$ such that if $\mathfrak{g}^j$ is the eigenspace for eigenvalue $j$ of $ad(h)$ then $x \in \mathfrak{g}^2$ and $\mathfrak{g}^2$ generates $\mathfrak{n}$ as a Lie algebra.

2. $ad(x) : \theta \mathfrak{n} \rightarrow \mathfrak{g}$ is injective.

To prove the first condition we show that there exists an $h$ with the appropriate properties. We will show that there exists $H \in \mathfrak{a}_o$ such that the eigenvalues of $ad(H)$ are integers and the eigenspace with eigenvalue 1 generates $\mathfrak{n}$. Let $\Phi(P_o, A_o)$ be the set of roots for $A_o$ on $\mathfrak{n}_o$. Then the weights of $A_o$ on $\mathfrak{n}$ are the elements of $\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$. Let $\Delta = \{\alpha_1, ..., \alpha_l\}$ denote the simple roots of $\Phi(P_o, A_o)$.
and (after possibly relabeling) \( \Delta_M = \{ \alpha_{r+1}, \ldots, \alpha_l \} \) the simple roots of \( \Phi(P_o \cap (MA), A_o) \). We prove

**Lemma 5.** If \( \alpha = \sum m_i \alpha_i \in \Phi(P_o, A_o) \) and \( \sum_{i \leq r} m_i \geq 2 \) then \( \alpha = \gamma + \delta \) with \( \gamma, \delta \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o) \).

Before we prove this lemma let us show how it implies the existence of the element \( H \). Denote by \( g^\alpha \) the root space for \( \alpha \in \Phi(P_o, A_o) \). We note that if \( \alpha, \beta \in \Phi(P_o, A_o) \) then \( [g^\alpha, g^\beta] = g^{\alpha + \beta} \) if \( \alpha + \beta \in \Phi(P_o, A_o) \) and 0 otherwise. So the lemma plus this observation imply that \( g^\alpha \subset [n, n] \) if and only if \( \alpha = \sum m_i \alpha_i \) with \( \sum_{i \leq r} m_i \geq 2 \). This implies that \( n \) is generated by the sum of the \( g^\alpha \) with \( \alpha = \sum m_i \alpha_i \) and \( \sum_{i \leq r} m_i = 1 \). Let \( h_1, \ldots, h_l \in a_o \) be defined by \( \alpha_i(h_j) = \delta_{ij} \). Set \( H = h_1 + \ldots + h_r \). Then \( H \in a \) has the desired properties.

We will now prove the lemma. We prove it by induction on \( |\alpha| = \sum_{i \leq l} m_i \). If \( |\alpha| = 2 \) then \( \alpha = \alpha_i + \alpha_j \) with \( i, j \leq r \) as desired. We now assume the result for \( 2 \leq |\alpha| \leq s \) and consider the case when \( |\alpha| = s+1 \). There exists \( i \) such that \( (\alpha, \alpha_i) > 0 \) (since otherwise \( (\alpha, \alpha) \leq 0 \)). Thus since \( \sum_{i \leq r} m_i \geq 2 \), \( \alpha - \alpha_i \) is in \( \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o) \). If \( i \leq r \) then we take \( \gamma = \alpha - \alpha_i \) and \( \delta = \alpha_i \). Otherwise, \( |\alpha - \alpha_i| = s \) and the sum of the coefficients of the \( \alpha_i \) with \( i \leq r \) hasn't changed. Hence the inductive hypothesis implies that \( \alpha - \alpha_i = \gamma + \delta \) with \( \gamma, \delta \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o) \). We therefore have \( [g^\gamma, g^\delta] = g^{\alpha - \alpha_i} \). Also \( [g^{\alpha - \alpha_i}, g^{\alpha_i}] = g^\alpha \). So applying the Jacobi identity we have

\[
g^\alpha = [[g^\gamma, g^\delta], g^{\alpha_i}] = [[g^\gamma, g^{\alpha_i}], g^\delta] + [g^\gamma, [g^\delta, g^{\alpha_i}]].
\]

Hence, \( \gamma + \alpha_i \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o) \) or \( \delta + \alpha_i \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o) \). Assume the latter, then \( \alpha = \gamma + (\delta + \alpha_i) \) which is the desired form.

Let \( h = 2H \). Then the space \( \{ X \in g | [h, X] = 2X \} \) generates \( n \). We set \( g' = \{ X \in g | [h, X] = jX \} \). Then we see that \( [\theta n, \theta n] = \sum_{j>1} g^{-2j} \). Thus since we are assuming that \( B(x, [\theta n, \theta n]) = 0 \) we have \( [h, x] = 2x \). Thus (1) above is satisfied. As for (2) we have

**Lemma 6.** If (1) above is satisfied then (2) is equivalent with the condition \( [p, x] = n \).

**Proof.** We note that

\[
(\theta n)^x = \ker \text{ad}(x)|_{\theta n} = \{ y \in \theta n | B(y, [x, g]) = 0 \}.
\]

Thus if \( [p, x] = n \) then since the pairing between \( n \) and \( \theta n \) is perfect we see that \( (\theta n)^x = 0 \). Assume that \( (\theta n)^x = 0 \). Noting that \( [x, g] \)
is $\text{ad}(h)$-invariant and that $[x, \theta n] \subset m \oplus \theta n$ we see that the displayed formula implies that $[p, x] \supset n$. Since $[p, x] \subset n$ the lemma follows. □

We can now give a reformulation of Theorem 3.4 of [Wa1]. We will use the notation

$$(M \otimes \mathbb{C}_{-\psi})^{\theta n} = \{ m \in M | (Y - \psi(Y))m = 0, Y \in \theta n \}.$$ 

**Theorem 7.** Assume that $\text{Ad}(P)x$ is open in $n$. Let $F$ be a finite dimensional representation of $\mathfrak{g}$. Then there exists an element $\Gamma \in U(\mathfrak{g} \mathbb{C}) \otimes \text{End}(F)$ depending only on $x$ and $F$ such that if $M \in W_\psi$ then $\Gamma : (M \otimes F) \otimes (\mathbb{C}_{-\psi})^{\theta n} \to (M \otimes F) \otimes \mathbb{C}_{-\psi}$ is a linear bijection. (Here $U(\mathfrak{g} \mathbb{C})$ is acting by the tensor product action on $M \otimes F$ and $\text{End}(F)$ only on $F$.)

4. An application of a theorem of Kolk and Varadarajan.

We maintain the notation of the previous sections. We set $W_G(A_o)$ equal to the Weyl group of $G$ with respect to $A_o$. For each $w \in W_G(A_o)$ we fix $w^* \in K$, normalizing $a_o$, such that $w = \text{Ad}(w^*)|_{a_o}$. Also let $w_M$ be the element of $W_{MA}(A_o)$ such that $w_M(\Phi((P_o \cap M)A, A_o)) = -\Phi((P_o \cap M)A, A_o)$. We can choose $w^*_M \in K \cap M$. Our application of the result of Kolk-Varadarajan rests on

**Lemma 8.** Let $h \in a$ be as in (1) in the previous section. Assume that $x \in n$ is such that $[h, x] = 2x$ and that $\text{Ad}(MA)x$ is open in $\mathfrak{g}^2$. If $w \in W_G(A_o)$ is such that $w\Phi(P_o, A_o) \supset \Phi((P_o \cap M)A, A_o)$ is such that

$$B(\text{Ad}(w^*)^{-1}\text{Ad}(w^*_M)n_o \cap \theta n, x) = 0$$

then $w = 1$.

**Proof.** The condition that

$$B(\text{Ad}(w^*)^{-1}\text{Ad}(w^*_M)n_o \cap \theta n, x) = 0$$

can be interpreted to mean that if $\alpha \in w_M\Phi(P_o, A_o)$ and

$$w^{-1}\alpha \in -(\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o))$$

then $w^{-1}\alpha(h) < -2$. This implies that if $\beta \in \Phi(P_o, A_o)$ and $\beta(h) = -2$ then $w\beta \notin w_M\Phi(P_o, A_o)$. Hence $w\beta \in -w_M\Phi(P_o, A_o)$. But $w\Phi(P_o, A_o)$ contains $\Phi(P_o \cap (MA), A_o)$ so this implies that

$$w\beta \in -(\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)).$$
Now $g^{-2}$ generates $\theta n$. Hence we have

$$w(\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)) \subset (\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o))$$

and our assumption on $w$ now implies that $w\Phi(P_o, A_o) = \Phi(P_o, A_o)$

this implies that $w = 1$. □

We will now apply the results of [KV]. We note that $P\overline{N}$ is open in $G$. We will denote by $U_{P,\sigma,\nu}$ the space of all $f \in I_{P,\sigma,\nu}^\infty$ such that $\text{supp}(f) \subset P\overline{N}$ is compact modulo $P$. The key result is

**Theorem 9.** Let $\chi$ be a character of $\overline{N}$ such that $d\chi(Y) = IB(x, Y)$

and $x$ satisfies the hypothesis of the previous lemma. Let $(\sigma, H_\sigma)$ be an irreducible finite dimensional representation of $M$. If $\lambda \in Wh_\chi(I_{P,\sigma,\nu}^\infty)$ and if $\lambda|_{U_{P,\sigma,\nu}} = 0$ then $\lambda = 0$.

**Proof.** We will be applying Theorem 3.15 p. 82 of [KV] case iii). However, we will be reversing the roles of right and left in their result. We consider the group $H = P \times \overline{N}$ acting on $G$ by $(p, y) \cdot x = pxy^{-1}$.

Then if

$$\Sigma = \{w \in W_G(A_o)| w\Phi(P_o, A_o) \supset \Phi(P_o \cap (MA), A_o)\}$$

then $G$ is the disjoint union $\cup_{w \in \Sigma} P \overline{N}$.

We consider the subgroup $H' = P \times \overline{N}$ acting on $G$ be the restriction of the action of $H$. This group has one open orbit which is the same as the unique open orbit of $H$, that is $H \cdot 1 = P\overline{N} = P\overline{N} = H' \cdot 1$. The other orbits of $H'$ are given as $H' \cdot w^*m$ with $m \in \overline{N} \cap M$. We assume that $w \neq 1$ and $m \in \overline{N} \cap M$ we will calculate the stabilizer of $w^*m$ in $H'$. We are looking at the set of all $p, y$ with $p \in P$ and $y \in \overline{N}$ such that $pw^*my^{-1} = w^*m$. That is $m^{-1}(w^*)^{-1}pw^*m = y$. Since $\overline{N} \cap M$ normalizes $\overline{N}$. We are therefore looking at pairs $(p, z)$ with $z \in \overline{N}$ and $z = (w^*)^{-1}pw^*$ with $p \in P$. Observe that the action of $Ad(w^*)$ stabilizes $a_o$ and permutes the root spaces. By the definition of $\Sigma$ we see that $z \in (w^*)^{-1}w_M^*N_o w_M^*w^* \cap \overline{N}$. Thus the projection into the second factor of the stabilizer of $w^*m$ has Lie algebra $Ad(w^*)^{-1}Ad(w^*_M)N_o \cap \theta n$. The previous lemma now implies that the character $\chi$ is not identically equal to 1 on $(w^*)^{-1}w_M^*N_o w_M^*w^* \cap \overline{N}$ and since the elements of this group act unipotently under $\sigma$ and in the adjoint representation we see that equation (3.27) in [KV] page 82 is satisfied (with right and left interchanged). To complete the proof starting with the conclusion of Theorem 3.15 in [KV] one uses standard Bruhat theoretic arguments (now using the finite number of orbits of $H$ on $G$). We will now sketch what is necessary to implement the method.

Let $\lambda \in Wh_\chi(I_{P,\sigma,\nu}^\infty)$ and assume that $\lambda|_{U_{P,\sigma,\nu}} = 0$. We label the elements of $\Sigma$ as $\{w_1, \ldots, w_r\}$ such that $X_j = \cup_{i \geq j} H \cdot w_i^*$ is closed in $G$.
We define $T$, a distribution on $G$ with values in $H_\sigma$ as follows: Let $f$ be a smooth compactly supported function with values in $H_\sigma$. Set

$$f^\nu(g) = \int_{M \times A \times N} f(ma)\sigma(m)^{-1}a^{-\nu-\rho}dmdadn$$

here we have made a choice of bi-invariant measure for each of the indicated groups. Then $f^\nu \in I_{P,\sigma,\nu}$. We set $T(f) = \lambda(f^\nu)$. Our assumption on $\lambda$ says that $\text{supp} T \subset X_2$. Suppose that we have shown that $\text{supp} T \subset X_j$, $j \geq 2$. Now, $H \cdot w_j^*$ is open in $X_j$ hence there exists $U$ open in $G$ such that $U$ is $H$-invariant and $U \cap X_j = H \cdot w_j^*$. Our assumption now implies that $\text{supp} T|_U \subset H \cdot w_j^*$. We can now apply Theorem 3.15 in [KV] to see that $T|_U = 0$. Thus $\text{supp} T \subset X_{j+1}$. Since $X_{r+1} = \emptyset$ the theorem follows.

5. The proof of the main theorem.

We will prove Theorem 2 of the introduction in this section. We first need a lemma which is essentially the same as Corollary 2 of [Wa3]. We use the notation in the introduction, in particular the action of $M_\chi$ on $Wh_\chi(I_{P,\sigma,\nu}^\infty)$

**Lemma 10.** Assume that $\chi$ is as in Theorem 9 and that $(\sigma, H_\sigma)$ is finite dimensional. Then $Wh_\chi(I_{P,\sigma,\nu}^\infty)$ is equivalent with a subrepresentation of the representation contragredient to $(\sigma|_{M_\psi}, H_\sigma)$ as an $M_\chi$-module.

**Proof.** We first note that if $T \in (C_c^\infty(N))^\prime$ (continuous linear functionals in the usual topology) is such that $T \circ R_x = \chi(x)^{-1}T(R_x f(y) = f(yx))$. Then

$$T(f) = c(T) \int_N f(\overline{n})\chi(\overline{n})dn.$$

This can be seen as follows. Let $\{X_j\}$ be a basis of $\text{Lie}(N)$ (thought of as left invariant vector fields). Then we have

$$X_j T = -d\chi(X_j)T$$

in the sense of distributions. Thus

$$\left(\sum_j X_j^2 - \sum_j (d\chi(X_j)^2)T = 0.$$

The elliptic regularity theorem implies that $T$ is given by integration against real analytic dim $N$-form on $\overline{N}$. The transformation law now easily implies that the form is $\chi(\overline{n})\omega$ with $\omega$ invariant.

If $f \in C_c^\infty(\overline{N})$ and $v \in H_\sigma$ then we define $S(f \otimes v)(p\overline{n}) = \sigma_\nu(p)f(\overline{n})v$ for $p \in P$, $\overline{n} \in \overline{N}$. If $\lambda \in Wh_\chi(I_{P,\sigma,\nu}^\infty)$ then we have $T_{\lambda,\nu}(f) =$
\[ \lambda(S(f \boxtimes v)) \] defines a distribution on \( \mathcal{N} \) and \( T_{\lambda,v}(R_x f) = \chi(x)T_{\lambda,v}(f) \) for all \( x \in \mathcal{N} \). Thus we have a \( \mathbb{C} \)-bilinear pairing defined on \( \text{Wh}_\chi(I_{P,\sigma,\nu}^\infty) \times H_\sigma \) defined by
\[
T_{\lambda,v}(f) = \langle \lambda, v \rangle \int_{\mathcal{N}} f(\overline{n}) \chi(n) d\overline{n}.
\]
A direct calculation shows that
\[
\langle m \cdot \lambda, v \rangle = \langle \lambda, \sigma(m)^{-1}v \rangle.
\]
These observations combined with Theorem 9 complete the proof. \( \square \)

We are now ready to begin in earnest to prove the main theorem. We first observe the following

**Proposition 11.** Let \( (\sigma, H_\sigma) \) be a smooth Fréchet representation of \( M \) of moderate growth then there exits a constant \( C_\sigma \) such that if \( \Omega_\sigma = \{ \nu \in a_\mathcal{N}^* | \text{Re}(\nu, \alpha) > C_\sigma, \alpha \in \Phi(P,A) \} \) then if \( \chi \) is a unitary character of \( \mathcal{N} \) then
\[
J_{P,\sigma,\nu}^\chi(f) = \int_{\mathcal{N}} f_{P,\sigma,\nu}(\overline{n}) \chi(\overline{n})^{-1} d\overline{n}
\]
converges absolutely and uniformly in compacta of \( \Omega_\sigma \).

This result is well known see for example Proposition 7.1 in [Wa1] (or putting in appropriate seminorms it is an easy consequence of Lemma 4.A.2.3 in [Wa2]).

Using this result and Lemma 10 it is easy to prove

**Proposition 12.** Let \( \chi \) be a generic character of \( \mathcal{N} \) and \( \nu \in \Omega_\sigma \) and assume that \( \dim H_\sigma < \infty \). Then
1. \( \text{Wh}_\chi(I_{P,\sigma,\nu}^\infty) = \{ \lambda \circ J_{P,\sigma,\nu}^\chi | \lambda \in (H_\sigma)^* \} \).
2. As an \( M_\chi \)-module \( \text{Wh}_\chi(I_{P,\sigma,\nu}^\infty) \) is equivalent with the contragradient of \( (\sigma_{|M_\chi}, H_\sigma) \).

With all of this at hand the proof of Theorem 2 is essentially same as the proof of Theorem 7.2 in [Wa1] since the notation in that paper is different, there are several misprints in that proof and perhaps too many details were left to the reader we will give the argument for the sake of clarity. Set \( \psi = d\chi \). If \( (\pi, V) \) is a smooth Fréchet representation of \( G \) then we set
\[
V'[\psi] = \{ T \in V' | \text{if } y \in \theta n \text{ there exists } k \text{ such that } (y - \psi(y))^k T = 0 \}.
\]
Then \( V'[\psi] \in \mathcal{W}_\psi \). If \( F \) is a finite dimensional representation of \( G \) then since \( \theta n \) acts nilpotently on \( F \) it is clear that
\[
(V \boxtimes F)'[\psi] = V'[\psi] \boxtimes F.
\]
Let \((\mu, F)\) be a finite dimensional irreducible representation of \(G\) such that

a) Setting \(F^{\theta_n} = \{v \in F | yv = 0, y \in \theta_n\}\) then \(M\) acts trivially on \(F^{\theta_n}\).

This implies that \(\dim F^{\theta_n} = 1\). This one dimensional space is invariant under \(a\) which therefore acts by a linear functional which we will denote \(-\Lambda\). We also assume

b) If \(\alpha \in \Phi(P, A)\) then \((\Lambda, \alpha) > 0\).

Such a representation always exists. Indeed, if \(d = \dim \mathfrak{n}\) then we can take for \(F\) the span of \(\bigwedge^d \text{Ad}(G) \cdot \bigwedge^d \mathfrak{n}\) in \(\bigwedge^d \mathfrak{g}\).

We set \(\sigma_v(\text{man}) = a^{\nu+\rho} \sigma(m)\). Then \((\sigma_v, H_\sigma)\) is a finite dimensional representation of \(P\). If \((\xi, H_\xi)\) is a finite dimensional continuous (hence smooth) representation of \(P\) then we set \(I_{P,\xi}^\infty\) equal to the space of all smooth \(f\) from \(G\) to \(H_\xi\) such that \(f(pg) = \xi(p)f(g), g \in G, p \in P\). \(G\) acts on \(I_{P,\xi}^\infty\) by the right regular action. We endow the space \(I_{P,\xi}^\infty\) with the \(C^\infty\) topology. We also note that the map \(f \mapsto f_{iK}\) defines an isomorphism of topological vector spaces between \(I_{P,\xi}^\infty\) and the space \(I_{iK \cap P}^\infty\) consisting of the smooth maps from \(K\) to \(H_\xi\) such that \(f(mk) = \xi(m)f(k)\) for \(m \in K \cap P\) and \(k \in K\) with the \(C^\infty\) topology. Fix a norm on \(H_\xi; ||...||\). Since \(K\) is compact we see that this topology can be defined using the seminorms \(p_x(f) = \sup_k ||xf(k)||\) for \(x \in U(\text{Lie}(K))\).

We now observe that if we put on \(I_{P,\xi}^\infty \otimes F\) the Fréchet space structure gotten by choosing a basis of \(F\) and looking at \(I_{P,\xi}^\infty \otimes F\) as a direct sum of \(\dim F\) copies of \(I_{P,\xi}^\infty\) then the map \(T(f \otimes v)(g) = f(g) \otimes \mu(g)v\) defines a continuous isomorphism between the smooth Fréchet representations \(I_{P,\xi}^\infty \otimes F\) and \(I_{P,\xi}^\infty \otimes \mu\). Let

\[H_\xi \otimes F = W_0 \supset W_1 \supset \ldots \supset W_r \supset W_{r+1} = \{0\}\]

Be a Jordan-Hölder series. Then \(W_i/W_{i+1}\) is equivalent with \(((\sigma_i)_{\nu_i}, H_{\sigma_i})\) for \(i = 0, \ldots, r\). This leads to the composition series

\[I_{P,\xi}^\infty \otimes F = V_0 \supset V_1 \supset \ldots \supset V_r \supset V_{r+1} = \{0\}\]

with \(V_i\) a closed smooth Fréchet subrepresentation of \(V_{i-1}\) and \(V_i/V_{i-1}\) is topologically equivalent with \(I_{P,\sigma_i,\nu_i}^\infty\). We will now apply these observations to the case when \(\xi = \sigma\). We note that in this case we can assume that \(\sigma_0 = \sigma\) and \(\nu_0 = \nu - \Lambda\). We assume that \(\nu\) is such that

\(\dim Wh_{\xi}(I_{P,\sigma,\nu}^\infty) = \dim H_{\sigma}\)

Then Theorem 7 implies that

\(\dim((I_{P,\sigma,\nu}^\infty(\nu) \otimes F) \otimes \mathbb{C}_{-\psi})^{\theta_n} = \dim H_{\sigma}\dim F\).
We have
\[(\langle I_{\rho,\omega}^{\infty}[\psi] \otimes F \otimes C_\psi \rangle \otimes V_r)^{\otimes n}|_{V_r} \subset V_{r'}^{(\psi)^{\otimes n}}\]
and since \(V_r\) is topologically isomorphic with \(I_{\rho,\omega,\nu}^{\infty}\) Lemma 9 implies that \(\dim V_{r'}^{(\psi)^{\otimes n}} \leq \dim H_{\sigma_r}\). We put
\[Z^j = \{\lambda \in \langle (I_{\rho,\omega}^{\infty}[\psi] \otimes F \otimes C_\psi \rangle \otimes V_r)^{\otimes n}|_{\lambda}|_{V_r} = 0\}\]
We therefore have \(\dim Z^r \geq \dim H_{\sigma_r} \dim F - \dim H_{\sigma_r}\). Now \(V_{r-1}/V_r\) is equivalent with \(I_{\rho,\omega,\nu}^{\infty}\) hence we see that
\[\dim Z_{r-1} \geq \dim H_{\sigma_r} \dim F - \dim H_{\sigma_r} - \dim H_{\sigma_{r-1}}\]
Continuing in this way we find that
\[\dim Z_1 \geq \dim H_{\sigma_r} \dim F - \sum_{i \geq 1} \dim H_{\sigma_i} = \dim H_{\sigma}\]
We note that \(Z_1\) injects into \((V_0/V_1)^{(\psi)^{\otimes n}}\) and since \(\dim (V_0/V_1)^{(\psi)^{\otimes n}} = \dim Wh_\chi(I_{\rho,\omega,\nu}^{\infty})\) we have shown that \(\dim Wh_\chi(I_{\rho,\omega,\nu}^{\infty}) \geq \dim H_{\sigma_r}\).
Since Lemma 9 asserts the reverse inequality we have proved
I) If \(\dim Wh_\chi(I_{\rho,\omega,\nu}^{\infty}) = \dim H_{\sigma}\) then \(\dim Wh_\chi(I_{\rho,\omega,\nu}^{\infty}) = \dim H_{\sigma}\)
Our hypothesis on \(\Lambda\) implies that if \(\nu\) is given then there exists \(k > 0\) with \(k \in \mathbb{Z}\) such that \(\nu + k\Lambda \in \Omega_{\sigma,\nu}\). Hence I) implies
II) \(\dim Wh_\chi(I_{\rho,\omega,\nu}^{\infty}) = \dim H_{\sigma}\) for all irreducible finite dimensional representations \((\sigma, H_{\sigma})\) of \(M\) and all \(\nu \in \mathfrak{a}_C^*\).
We note that at this point we have proved part 3. of Theorem 2. We will now complete the proof of parts 1. and 2. We define \(\delta : I_{\rho,\omega}^{\infty} \otimes F \rightarrow I_{\rho,\omega}^{\infty}\) by \(\delta(f \otimes \mu)(k) = f(k) \otimes \mu(k)\). Let \(q\) denote the projection of \(F\) onto \(F/nF\) then we can form \(T_\Lambda = (I \otimes q) \circ \delta\).
Then \(T_\Lambda : I_{\rho,\omega}^{\infty} \otimes F \rightarrow I_{\rho,\omega}^{\infty}\) is surjective continuous and
\[T_\Lambda \circ (\pi_{\rho,\omega,\nu}(g) \otimes \mu(g)) = \pi_{\rho,\omega,\nu}(g) \circ T_\Lambda(*)\]
Note that \(T_\Lambda\) only depends on \(\sigma\) and \(F\) and not on \(\nu\). Now suppose that we have proved
III) The map \(\nu \mapsto J_{\rho,\omega}^{\infty}(f)\) extends to a holomorphic map of \(U_t = \{\nu \in \mathfrak{a}_C^*|(\alpha, \nu) > t, \alpha \in \Phi(P, A)\}\) to \(H_{\sigma}\).
and
IV) If \(\nu \in U_t\) and \(\eta \in Wh_\chi(I_{\rho,\omega,\nu}^{\infty})\) then there exists \(\lambda \in H_{\sigma}^*\) such that \(\eta = \lambda \circ J_{\rho,\omega,\nu}^{\infty}\).
We set $\pi_\nu = \pi_{P,\sigma,\nu}$, $J_\nu = J_{P,\sigma,\nu}$. Fix $\lambda_1, \ldots, \lambda_m$ a basis of $H_\sigma^*$ and let $\xi_1, \ldots, \xi_d$ be a basis of $F^*$. If $\nu \in U_t$ then set $\gamma_i(\nu) = \lambda_i \circ J_\nu$. Then Theorem 7 says that there exist $p^{k}_{ij} \in U(g)$ and $A^{k}_{ij} \in \text{End}(F^*)$ with $1 \leq i \leq m, 1 \leq j \leq d, 1 \leq k \leq dm$ such that the elements
\[
\zeta_k(\nu) = \sum_{i,j}^k p^{k}_{ij}(\gamma_i(\nu) \otimes \Lambda^k_{ij})
\]
form a basis of $(((\mathbb{C}^\infty(\nu) \otimes F^*) \otimes \mathbb{C}_{-\nu})^{\sigma})^m$. Fix $\nu_o \in U_t$ and set $Z_\Lambda = \text{ker} T_\Lambda$ (not that $Z_\Lambda$ depends only on $\sigma$ and $F$ but not on $\nu$). Then the argument proving II) above implies that
\[
\dim(\bigoplus_{i \leq dm} \mathbb{C} \zeta_i(\nu_o))|_{Z_\Lambda} = dm - m.
\]
We may assume after relabeling that $\zeta_{m+1}(\nu_o)|_{Z_\Lambda}, \ldots, \zeta_{dm}(\nu_o)|_{Z_\Lambda}$ are linearly independent. Thus there is an open neighborhood, $W$, of $\nu_o$ such that $\zeta_{m+1}(\nu)|_{Z_\Lambda}, \ldots, \zeta_{dm}(\nu)|_{Z_\Lambda}$ are linearly independent for $\nu \in W$. We therefore see that there exist holomorphic functions $a_{ij}$, $1 \leq i \leq m, m+1 \leq j \leq dm$ on $W$ such that
\[
\zeta_i(\nu)|_{Z_\Lambda} = \sum_{j > m} a_{ij}(\nu) \zeta_j(\nu)|_{Z_\Lambda}, \nu \in W.
\]
If we set
\[
\phi_i(\nu) = \zeta_i(\nu) - \sum_{j > m} a_{ij}(\nu) \zeta_j(\nu), 1 \leq i \leq m, \nu \in W
\]
then we note that $\phi_1(\nu), \ldots, \phi_m(\nu)$ are linearly independent and vanish on $Z_\Lambda$. Now
\[
T_\Lambda : (I^\infty_{\sigma[K \cap \rho]} \otimes F)/Z_\Lambda \to I^\infty_{\sigma[P \cap K]}
\]
is an isomorphism of topological vector spaces. Thus $(*)$ above implies that $\phi_1(\nu), \ldots, \phi_m(\nu)$ are holomorphic in on $W$ with values in $(I^\infty_{\sigma[K \cap \rho]})'$ and
\[
\phi_i(\nu)(\pi_{\nu-\Lambda}(\pi)) = \chi(\pi)^{-1} \phi_i(\nu)(f), \pi \in \overline{N}, f \in I^\infty_{\sigma[K \cap \rho]}.
\]
If $f \in C(\overline{N})$ and $h \in H_\sigma$ then set
\[
f_{\nu,h}(mam\overline{n}) = f(\overline{n})a^{\nu+\rho}(m)h.
\]
Then the $f_{\nu,h}$ span the space $U_{\sigma,\nu}$. We argue as in the proof of Lemma 9 to see that if we set $\Phi(f) = \int_{\overline{N}} \chi(\overline{n})^{-1} f(\overline{n})d\overline{n}$ then
\[
\phi_i(\nu)(f_{\nu,h}) = \alpha_i(\nu)(h)\Phi(f)
\]
with $\alpha_1(\nu), \ldots, \alpha_m(\nu)$ holomorphic on $W$ with values in $H_\sigma^*$ and for each $\nu \in W$ giving a basis of $H_\sigma^*$. Thus there exist holomorphic functions
\[ b_{ij}(\nu) \text{ on } W \text{ such that} \]
\[ \mu_i = \sum_j b_{ji}(\nu) \alpha_j(\nu), i = 1, \ldots, m. \]

Hence if
\[ \omega_i(\nu) = \sum_j b_{ji}(\nu) \phi_j(\nu), i = 1, \ldots, m \]
then
\[ \omega_i(\nu)(f) = \mu_i \left( \int \chi(\mu) f(\mu) d\mu \right) \]
for \( f \in U_{\sigma, \nu - \Lambda} \). At this point we have proved that there exist \( \omega_i(\nu) \) holomorphic on \( U_t \) with values in \( I_{\sigma, \nu - \Lambda}^\infty \), such that

V) \( \omega_i(\nu) \in Wh(\mu_I(\Lambda, \alpha)), i = 1, \ldots, m \) and are linearly independent.

VI) \( \omega_i(\nu)(f) = \omega_i(\nu)(f) = \mu_i \left( \int \chi(\mu) f(\mu) d\mu \right) \) for \( f \in U_{\sigma, \nu - \Lambda} \).

The uniqueness statement in Theorem 7 implies that if \( \nu - \Lambda \in U_t \) then \( \omega_i(\nu) = \mu_i \circ J_{\nu - \Lambda} \). Set \( q = \min_{\alpha \in \Phi(P, A)} (\Lambda, \alpha) \). Then \( q > 0 \). We have proved that III) and IV) are true with \( t \) replaced by \( t - q \). Since \( U_{t - rq} \subset U_{t - (r + 1)q} \) and \( \cup_{r > 0} U_{t - rq} = \mathfrak{a}^*_C \) we have finally completed the proof of Theorem 2.

We will now prove Corollary 3. Let \( k \) be as in the statement. Then if we define \( \eta(\mu) = \chi(k^{-1} \mu) \) we have
\[ J^X_{\nu, \sigma, \nu}(f) = J^o_{\nu, \sigma, \nu}(\pi_{\nu, \sigma, \nu}(k) f). \]

Here the difference between the “\( J \)” on the left hand side and on the right is that they involve characters for different unipotent groups. Thus to prove the Corollary we must show that \( Ad(k)y \) is a Richardson element. We set \( e_o = Ad(k)y \), \( f_o = Ad(k)x \) and \( h_o = -Ad(k)h \) then \( [e_o, f_o] = h_o, [h_o, e_o] = 2e_o, [h_o, f_o] = -2f_o \). From the theory of representations of \( \mathfrak{sl}(2, \mathbb{C}) \) one sees that \( ad(e_o) \mathfrak{g}^r = \mathfrak{g}^{r + 2} \) (here the \( \mathfrak{g}^r \) are for \( h_o \)). This implies that \( ad(e_o) \left( \sum_{i \geq 0} \mathfrak{g}^{2r} \right) = \left( \sum_{i \geq 1} \mathfrak{g}^{2r} \right) \). Thus \( ad(Lie(P))e_o = \mathfrak{n} \). This implies the assertion.
References


University of California, San Diego
E-mail address: nwallach@ucsd.edu