1. Express the following rational numbers (given in base 10) as Egyptian fractions and to 2 digits in sexagesimal.
   a) \( \frac{3}{7} \)  b) \( \frac{5}{11} \).

2. a) The Babylonians had a general method of solving systems of the form
   \[
   x + y = a \\
   xy = b.
   \]
   They thought of \( x = \frac{a}{2} + z \) and \( y = \frac{a}{2} - z \). Then deduced that
   \[
   x = \frac{a}{2} + \sqrt{\left( \frac{a}{2} \right)^2 - b} \\
   y = \frac{a}{2} - \sqrt{\left( \frac{a}{2} \right)^2 - b}.
   \]
   Explain this.

   b) If the problem were of the form
   \[
   x + y = a \\
   xy + x - y = b.
   \]
   Then they would make the substitution \( y = z - 2 \). Show that the system becomes
   \[
   x + z = a + 2 \\
   xz = a + b.
   \]

   c) Use the above to solve the following Babylonian problem:
   To the area of a rectangle the excess of the length to the width is added giving 120 [This is in base 10.] moreover the sum of the length and width is 24. Find the dimensions of the rectangle.

3. Recall that a Pythagorean triple is three positive integers \( a, b, c \) with \( a^2 + b^2 = c^2 \). At least one of \( a \) and \( b \) must be even. Why?

4. What are the divisors of 40? In general what are the divisors of \( 2^k5^t \) for \( k = 1, 2, 3, \ldots \)?

5. Recall that the Fibonacci sequence is given by \( F_0 = F_1 = 1, F_{k+1} = F_k + F_{k-1}, k \geq 1 \).
a) Show that \( F_n^2 = F_n F_{n+1} - F_n F_{n-1} \).

b) Use this to show that \( F_0^2 + F_1^2 + \ldots + F_n^2 = F_n F_{n+1} \). (Hint: Use induction.)

6. a) Show that if \( k > 0 \) then the last digit of \( 16^k \) to base 10 is 6. (Hint: A number \( n \) has last digit 6 to base 10 if and only if \( n = 6 + 10a \) with \( a \) a positive integer.)

b) Show that an even perfect number has last digit to base 10, 6 or 8. (Hint: Write the number as \( (2^p - 1)2^{p-1} \) with \( p \) a prime. If \( p = 2 \) the number is 6. If \( p > 2 \) the \( p = 4m + 1 \) or \( 4m + 3 \). In the first case the last digit is 6 in the second it is 8. To see this in the first case write
\[
(2^{4k+1} - 1)2^{4k} = 2 \cdot 16^{2k} - 16^k,
\]
in the second write
\[
(2^{4k+3} - 1)2^{4k+2} = 2 \cdot 16^{2k+1} - 4 \cdot 16^k.
\]
Now use part a).)

7. a) Consider the equation
\[
x^3 + bx^2 + cx + d = 0.
\]
If \( x \) is a rational solution show that \( x \) is an integer that divides \( d \). (Hint: Write \( x = \frac{u}{v} \) with \( u, v \) relatively prime and multiply through by \( v^3 \).)

b) Use this to find all roots of \( x^3 - 7x^2 + 4x + 24 \).

8. Use the Egyptian method of completing the square to solve the following problem (similar to one in the Moscow Papyrus):

The width of a rectangle is \( \frac{1}{3} + \frac{1}{15} \) of the length the area is 40. Find its dimensions.

9. Solve the following problems from the Algebra of Abu Kamil.

a) 10 Dinar is divided equally among a group of men so that when 6 men join the group and 40 is divided equally among them each receives the same amount as before. How many men were there?

b) Divide 10 into two parts (i.e. the parts add up to 10) in such a way that if 50 is divided by one part and 40 by the other and the two fractions are multiplied 125 will result. (Hint: Use problem 1 c).)

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10. Euclid’s method of solving quadratic equations was purely geometric. Consider figure 1. Here the line $AB$ is bisected at $P$. The line perpendicular to $AB$ at $B$ is constructed and extended to $E$ at length $b$. The circle of radius $PE$ is drawn an the radius containing $B$ is $PQ$. Euclid asserts that the rectangle of sides $AQ$ and $BQ$ has area equal to the square of side $BE (b^2)$. Show that this is true. What quadratic equation is Euclid solving? Show that if $b = a$ then $\frac{BQ}{AB}$ is the golden ratio.