Note Title 2/11/2009

$d > 0$, $d \in \mathbb{Z}$, d not a square.

$$\mathbb{R} \supset \mathbb{Q}(\sqrt{d}) = \{u + \sqrt{d}\, v \mid u, v \in \mathbb{Q}\}.$$

$$(u_1 + \sqrt{d}\, v_1)(u_2 + \sqrt{d}\, v_2)$$

$$= u_1 u_2 + d v_1 v_2 + (v_1 u_2 + u_1 v_2)\sqrt{d}.$$

Closed under addition, mult.

$u+\sqrt{d}\,v=0 \Rightarrow u=0$ and $v=0$

if $v=0$. then $u=0$ and

if $v\neq 0$ then $\sqrt{d}=-\frac{u}{v}\in\mathbb{Q}$.

But $\sqrt{d}$ is irrational $\Rightarrow\Leftarrow$

$u+\sqrt{d}\,v\neq 0 \Rightarrow u^2-dv^2\neq 0$

if $u^2-dv^2=0$ then $d=\frac{u^2}{v^2}=\left(\frac{u}{v}\right)^2$

$\Rightarrow \frac{u}{v}=\pm\sqrt{d}$.

$\mathbb{Q}(\sqrt{d}) \ni \frac{u-\sqrt{d}\,v}{u^2-dv^2}=\frac{1}{u+\sqrt{d}\,v}$

$\sigma: \mathbb{Q}(\sqrt{d}) \longrightarrow \mathbb{Q}(\sqrt{d})$

$\sigma(u+\sqrt{d}\,v) = u - \sqrt{d}\,v.$ (Conjugation

conjugate

$u, v \in \mathbb{Q}.$

$\sigma(\xi+\nu) = \sigma(\xi)+\sigma(\nu), \quad \sigma(\xi\cdot\nu) = \sigma(\xi)\sigma(\nu)$

$\sigma(1) = 1 \qquad \sigma(\xi^{-1}) = \sigma(\xi)^{-1}$ if $\xi \neq 0$

$(u_1+\sqrt{d}\,v_1)(u_2+\sqrt{d}\,v_2) =$

$u_1u_2 + d\,v_1v_2 + (v_1u_2+u_1v_2)\sqrt{d}.$

$\sigma(\qquad) =$

$u_1 u_2 + d v_1 v_2 - (v_1 u_2 + u_1 v_2)\sqrt{d}.$

$= \underline{(u_1 - \sqrt{d}\, v_1)(u_2 - \sqrt{d}\, v_2)} = \sigma(\xi)\sigma(\nu)$

<u>Theorem</u>. If $\alpha \in \mathbb{R}$ is a quadratic irrational then

$\alpha = [q_0, \ldots, q_r, \overline{u_1 \cdots u_t}]$ $\quad q_i \in \mathbb{Z}$

$q_1, \ldots, q_r > 0.$ $\quad u_i > 0 \quad u_i \in \mathbb{Z}$

Proof: $\alpha = [q_0, \underbrace{q_1, \ldots}_{q_i > 0}]$

If $\alpha' = [q_1, \ldots]$ then α' quadratic irrational and $\alpha' > 0$.

WLG (WLOG) may assume $\alpha > 0$.

$$\alpha = \frac{P_0 + \sqrt{d}}{Q_0}$$

$d \in \mathbb{Z}$ ($d > 0$, d not a square)

$Q_0 \mid (d - P_0^2)$

$\alpha_0 = \alpha, \quad q_0 = [\alpha_0]$

$P_{i+1} = q_i Q_i - P_i$ $\qquad Q_{i+1} \mid (d - P_{i+1}^2)$

$Q_{i+1} = (d - P_{i+1}^2)/Q_i$

$\alpha_{i+1} = \dfrac{P_{i+1} + \sqrt{d}}{Q_{i+1}}, \quad q_{i+1} = [\alpha_{i+1}].$

Must show eventually periodic.

$\alpha_{i+1} = \dfrac{1}{\alpha_i - q_i}.$

$\alpha_0 = \dfrac{a_{j-1}\alpha_j + a_{j-2}}{b_{j-1}\alpha_j + b_{j-2}}$ $\qquad \dfrac{a_j}{b_j}$ convergents

$$\begin{bmatrix} \alpha_{j-1} & a_{j-2} \\ b_{j-1} & b_{j-2} \end{bmatrix}^{-1} = (-1)^j \begin{bmatrix} b_{j-2} & -a_{j-2} \\ -b_{j-1} & a_{j-1} \end{bmatrix}$$

$$\Rightarrow \quad \frac{b_{j-2}\alpha_0 - a_{j-2}}{-b_{j-1}\alpha_0 + a_{j-1}} = \alpha_j .$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2x2 invertible ~~matrix~~ $\boxed{\text{Prove this}}$

$\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ inverse then if

$$w = \frac{az+b}{cz+d} \quad \text{then } z = \frac{rw+s}{tw+u}$$

$$\alpha_j = -\frac{b_{j-2}}{b_{j-1}}\left[\frac{\alpha_0 - \frac{a_{j-2}}{b_{j-2}}}{\alpha_0 - \frac{a_{j-1}}{b_{j-1}}}\right]$$

Hit both sides by σ.

$$\sigma(\alpha_j) = -\frac{b_{j-2}}{b_{j-1}}\left[\underbrace{\frac{\sigma(\alpha_0) - \frac{a_{j-2}}{b_{j-2}}}{\sigma(\alpha_0) - a_{j-1}/b_{j-1}}}\right]$$

$$\lim_{j\to\infty} \frac{\sigma(\alpha_0)-\alpha_0}{\sigma(\alpha_0)-\alpha_0} = 1$$

$\alpha_0 = u + \sqrt{d}\,V$

$\sigma(\alpha_0) = u - \sqrt{d}\,V$

$$\frac{-2\sqrt{d}V}{-2\sqrt{d}V} = 1.$$

If $j \geq n$ term in brackets is > 0

$b_{j-2}/b_{j-1} > 0$

$\Rightarrow \sigma(\alpha_j) < 0$ if $j \geq n$.

$$\Rightarrow \quad 0 < \alpha_j - \sigma(\alpha_j) = \frac{2\sqrt{d}}{Q_j} \Rightarrow Q_j > 0$$

for $j \geq n$.

$j \geq n$

$$0 < Q_{j+1} = \frac{d - P_{j+1}^2}{Q_j} \Rightarrow P_{j+1}^2 < d$$

$$Q_{j+1} < \frac{d}{Q_j} < d.$$

If $j > n$ then (P_j, Q_j) take only a finite number of values. and $t > 0$ $j, t \in \mathbb{Z}$

$\Rightarrow \exists\ j > n$ such that

$$(P_j, Q_j) = (P_{j+t}, Q_{j+t})$$

$$\alpha_j = \frac{P_j + \sqrt{d}}{Q_j} = \frac{P_{j+t} + \sqrt{d}}{Q_{j+t}} = \alpha_{j+t}$$

$q_j = \lceil \alpha_j \rceil$ $\qquad q_{j+t} = \lceil \alpha_{j+t} \rceil$

$$\Rightarrow P_{j+1} = q_j Q_j - P_j = P_{j+1+t}$$

$$Q_{j+1} = \frac{d - P_{j+1}^2}{Q_j} = Q_{j+1+t}$$

$$\alpha_{j+1} = \frac{P_{j+1} + \sqrt{d}}{Q_{j+1}} = \alpha_{j+1+t}$$

$$q_{j+1} = \lfloor \alpha_{j+1} \rfloor = \lfloor \alpha_{j+1+t} \rfloor = q_{j+1+t}$$

$$\Rightarrow \quad q_{j+k} = q_{j+k+t} \quad \text{for all}$$

$k = 0, 1, 2, \ldots$ QED.

suppose have $u_i, v_i \in \mathbb{Q}$.

$$u_1^2 - d v_1^2 = 1 \quad , \quad u_2^2 - d v_2^2 = 1$$

$$(u_1 + \sqrt{d}\, v_1)(u_2 + \sqrt{d}\, v_2) = u_3 + \sqrt{d}\, v_3 .$$

assert $\quad u_3^2 - d v_3^2 = 1$

multiply times conj

$$(u_1 + \sqrt{d}\, v_1)(u_2 + \sqrt{d}\, v_2)(u_1 - \sqrt{d}\, v_1)(u_2 - \sqrt{d}\, v_2)$$

$$= (u_1^2 - d v_1^2)(u_2^2 - d v_2^2) = 1$$

But also

$$= (u_3 + \sqrt{d}\, v_3)(u_3 - \sqrt{d}\, v_3) = u_3^2 - d v_3^2$$