

Practice Problems for the Midterm

1. Let $m \in \mathbb{Z}$ and $m > 1$. Show that if $a \in \mathbb{Z}$ and $\gcd(a, m) = \gcd(a - 1, m) = 1$ then

$$1 + a + a^2 + \dots + a^{\phi(m)-1} \equiv 0 \pmod{m}.$$

(Here $\phi(m)$ is Euler's ϕ -function, as usual.)

2. Let p be a prime. Recall that the principal Dirichlet character modulo p is given by

$$\chi_0(n) = \begin{cases} 1 & \text{if } p \nmid n \\ 0 & \text{if } p \mid n \end{cases}.$$

For what primes p does there exist a non-principal Dirichlet character χ such that $\chi^3 = \chi_0$?

3. Find the smallest prime p such that there is an integral solution to $x^2 - x - 1 \equiv 0 \pmod{p}$.

4. Let α be the number with simple continued fraction $[m, m, m, m, \dots]$ (i.e. $q_j = m$ all j). Write α in the form $\frac{P+\sqrt{d}}{Q}$ with $P, Q \in \mathbb{Z}$ and $Q \mid (d - P^2)$.

5. Let $d \in \mathbb{Z}$ be positive and not a square. Let $P_0 = 0, Q_0 = 1, \alpha_0 = \sqrt{d}, q_0 = [\alpha_0]$ and consider the recursion

$$P_{n+1} = Q_n q_n - P_n, Q_{n+1} = \frac{d - P_{n+1}^2}{Q_n}, \alpha_{n+1} = \frac{P_{n+1} + \sqrt{d}}{Q_{n+1}}, q_{n+1} = [\alpha_{n+1}].$$

Prove that there are an infinite number of n such that $Q_{2n} = 1$. (Hint: You should review the material on Pell's equation for this problem.)