

# On a Theorem of Milnor and Thom

*Nolan R. Wallach*

## 1. Introduction

In [M1],[T], Milnor and Thom (independently) proved an estimate on the sum of the Betti numbers (relative to an arbitrary field of coefficients) of the set of zeros of polynomials of degree at most  $k > 0$  in  $\mathbb{R}^n$ , essentially  $C_n k^n$  (Milnor's estimate is  $k(2k - 1)^{n-1}$ , Thom's is essentially twice Milnor's). In particular, this result gives a quantitative version of Whitney's (earlier) theorem [W], that says that the number of connected components is finite.

Most applications of the Theorem of Milnor and Thom are to the implied estimate on the number of connected components. For example, in [B] the estimate was used to determine lower bounds for the complexity of certain algebraic computation trees. One purpose of this article is to give a proof (following Milnor's methods) of the estimate on the number of connected components that uses only advanced calculus, elementary topology and Sard's theorem (the special cases of Sard's theorem that are used will also be sketched in this article) that should be accessible to mathematicians and computer scientists who are not experts in algebraic topology. Another is that the proof of Lemma 1 in [M1], left quite a bit to the reader. The first two sections of this article are devoted to an elementary proof of this lemma (see Theorem 3.4). We also give a less elementary proof in section 7 that gives a quantitative upper bound for the number of irreducible components of a variety over an algebraically closed field in terms of the degrees of a defining set of equations. This result may be of independent interest. Sections 7,8,9 involve more algebraic geometry and constitute whatever is new in this paper.

In [M1], Milnor indicates that he has no examples where  $C_n \neq 1$ . This suggests the problem of proving (or disproving) the contention that we can take  $C_n = 1$ . In section 8 we prove that if  $n = 2$  the answer (for the number of connected components) is affirmative. In section 9 we give an affirmative answer for non-singular hypersurfaces for the sum of the Betti numbers. This result gives a sharper upper bound for the sum of the Betti numbers of a set of the form  $\mathbb{R}^n - X$  where  $X$  is the zero set of a set of polynomials

with real coefficients.

This article is an outgrowth of lectures that the author gave on real algebraic geometry during a three quarter course in algebraic geometry at the University of California, San Diego. Beside graduate students in mathematics there were also regular participants from computer science and economics. We would like to thank R. Paturi for suggesting the Milnor-Thom theorem as a topic in the course and for his lectures on complexity related to [B].

Finally, this article is dedicated to the memory of my friend Joe D'Atri. His untimely death has left a void in the differential geometry community. His interests in and out of mathematics enriched all of our lives.

## 2. Generic finite varieties

Let  $k$  denote an algebraically closed field and let  $V$  denote an  $n$ -dimensional vector space over  $k$ . We use the notation  $\mathcal{P}^r(V)$  for the space of polynomial functions on  $V$  that are homogeneous of degree  $r$ . We set

$$W_{m_1, \dots, m_n} = \mathcal{P}^{m_1}(V) \times \dots \times \mathcal{P}^{m_n}(V).$$

We look upon  $W_{m_1, \dots, m_n}$  as an  $N = \sum \binom{m_i + n - 1}{n - 1}$  dimensional vector space over  $k$ . We fix  $m_i > 0$ ,  $i = 1, \dots, n$  and set  $W = W_{m_1, \dots, m_n}$ . If  $g \in W$  then we look upon  $g$  as both a polynomial map of  $k^n$  to  $k^n$  and an ordered set of polynomials. Set  $\mathcal{P}(V)$  equal to the algebra of polynomials on  $V$ . If  $g \in W$  then set  $I_g$  equal to the ideal generated by the entries of  $g$ .

We note that  $I_g$  is a graded subspace of  $\mathcal{P}(V)$ . Thus  $R_g = \mathcal{P}(V)/I_g$  inherits a natural grade. Set  $R_g^j$  equal to the  $j$ -th homogeneous component and  $h_g(t) = \sum_j t^j \dim R_g^j$  (thought of as a formal power series). We set

$$h(t) = \prod_{j=1}^n (1 + t + \dots + t^{m_j - 1}).$$

The following result is no doubt well known.

**Proposition 2.1.** *The set,  $\Omega_{m_1, \dots, m_n}$ , of all  $g \in W$  such that  $h_g(t) = h(t)$  is non-empty and Zariski open in  $W$ .*

We will need some notation before we give our (elementary) proof. Let  $Z$  be an  $n$ -dimensional vector space over  $k$  and let  $z_1, \dots, z_n$  be a basis of  $Z$ . We grade  $Z$  by setting  $\deg(z_i) = m_i$ . Then  $Z = \bigoplus Z^p$  with  $Z^p$  the span of the  $z_i$  with  $\deg(z_i) = p$ . We grade  $\mathcal{P}(V) \otimes Z$  by setting  $(\mathcal{P}(V) \otimes Z)^j =$

$\sum \mathcal{P}^{j-i}(V) \otimes Z^i$ . We define

$$\partial(g) : \mathcal{P}(V) \otimes Z \longrightarrow \mathcal{P}(V)$$

by

$$\partial(g)(f \otimes z_i) = fg_i$$

(here  $g = (g_1, \dots, g_n)$ ). Then  $\partial(g)(\mathcal{P}(V) \otimes Z)^j \subset \mathcal{P}^j(V)$  and  $\partial(g)(\mathcal{P}(V) \otimes Z) = I_g$ .

Define  $h_j \in \mathbb{Z}$  by

$$h(t) = \sum_j h_j t^j.$$

Set  $d = m_1 + \dots + m_n - n$ . Set  $p_j = \binom{j+n-1}{n-1} - h_j$ . It is easy to see that  $p_j \geq 0$ . For each  $1 \leq j \leq d+1$  choose bases of  $(\mathcal{P}(V) \otimes Z)^j$  and  $\mathcal{P}^j(V)$  and if  $p_j > 0$  let  $\Phi_{j,i}(g)$  be an enumeration of the  $p_j \times p_j$  minors of the restriction of  $\partial(g)$  to  $(\mathcal{P}(V) \otimes Z)^j$ . We set

$$\Omega^\circ = \{g \in W \mid \text{if } p_j > 0 \text{ there exists } i \text{ s.t. } \Phi_{j,i}(g) \neq 0\}.$$

It is clear that  $\Omega^\circ$  is a Zariski open subset. We also note that  $g = (x_1^{m_1}, \dots, x_n^{m_n}) \in \Omega^\circ$ . So  $\Omega^\circ$  is non-empty.

We note that  $I_g$  is a graded subspace of  $\mathcal{P}(V)$ . Thus  $R_g = \mathcal{P}(V)/I_g$  inherits a natural grade. Set  $R_g^j$  equal to the  $j$ -th homogeneous component and  $h_g(t) = \sum_j t^j \dim R_g^j$  (thought of as a formal power series).

**Lemma 2.2.** *If  $g \in \Omega^\circ$  then  $h_g = h$ .*

**Proof.** Fix  $g \in \Omega^\circ$ . Since  $h_{d+1} = 0$ ,

$$\partial(g)(\mathcal{P}(V) \otimes Z)^{d+1} = \mathcal{P}(V)^{d+1}.$$

Also, if  $0 \leq j \leq d$  then  $\dim R_g^j \leq \dim \mathcal{P}^j(V) - p_j = h_j$ . Let  $U$  be a graded subspace of  $\mathcal{P}(V)$  such that  $U \oplus I_g = \mathcal{P}(V)$ . It is easy to see that  $\dim U \cap \mathcal{P}^j(V) = \dim R_g^j$  and that if  $u_1, \dots, u_p$  is a basis of  $U$  then  $\sum k[g_1, \dots, g_n]u_i = \mathcal{P}(V)$ . Let  $w_1, \dots, w_n$  be indeterminates and grade  $k[w_1, \dots, w_n]$  by setting  $\deg w_i = m_i$ . Then we have a graded surjection,  $\Psi$ , of  $k[w_1, \dots, w_n] \otimes U$  to  $\mathcal{P}(V)$  given by  $f[w_1, \dots, w_n] \otimes u \mapsto f[g_1, \dots, g_n]u$ . Let  $Y = \ker \Psi$ . Then  $Y$  is graded and we have an identity of formal power series

$$\frac{h_g(t)}{\prod_{j=1}^n (1 - t^{m_j})} - \sum t^j \dim Y^j = \frac{1}{(1 - t)^n}.$$

Since

$$\frac{h(t)}{\prod_{j=1}^n (1 - t^{m_j})} = \frac{1}{(1 - t)^n},$$

we conclude that  $\dim R_g^j = h_j$ . That is  $h_g(t) = h(t)$ . This completes the proof of the Lemma.

If  $g \in W$  and if  $h_g(t) = h(t)$  then it is easily seen that  $\dim \partial(g)(\mathcal{P}(V) \otimes \mathbb{Z})^j = p_j$  for  $1 \leq j \leq d + 1$ . Hence  $g \in \Omega^o$ . Thus  $\Omega^o = \Omega$ .

In order to drop the condition of homogeneity we must recall some elementary facts about the relationship between filtrations and gradings. Let  $I$  be an ideal in  $\mathcal{P}(V)$ . Set  $R = \mathcal{P}(V)/I$ . We put  $\mathcal{P}_j(V) = \sum_{i \leq j} \mathcal{P}^i(V)$ . Let  $\pi$  denote the natural projection of  $\mathcal{P}(V)$  onto  $R$ . Set  $R_j = \pi(\mathcal{P}_j(V))$ . Then  $R_j \subset R_{j+1}$  and  $\cup R_j = R$ . Put (a usual)  $Gr^j R = R_j/R_{j-1}$  ( $R_{-1} = 0$ ) and  $GrR = \oplus Gr^j R$ . We use the obvious addition and multiplication on  $GrR$  to make it into a graded algebra over  $k$ .

If  $f \in \mathcal{P}_j(V)$ ,  $f \notin \mathcal{P}_{j-1}(V)$  then  $f = f_0 + \dots + f_j$  with  $f_i \in \mathcal{P}^i(V)$  and  $f_j \neq 0$ . We put  $f_{top} = f_j$ . Denote by  $I_{top}$  the linear span of the  $f_{top}$  for  $f \in I$ ,  $f \neq 0$ . Then it is easy to see that  $I_{top}$  is a graded ideal and that  $GrR$  is isomorphic with  $\mathcal{P}(V)/I_{top}$  as a graded algebra over  $k$ .

With this formalism in place we can state the main result of this section.

**Proposition 2.3.** *Let  $f_1, \dots, f_n \in \mathcal{P}(V)$  set  $g_i = (f_i)_{top}$  and assume that  $\deg g_i = m_i > 0$ . Let  $I$  be the ideal  $\sum \mathcal{P}(V)f_i$ . If  $g = (g_1, \dots, g_n) \in \Omega_{m_1, \dots, m_n}$  the algebraic set  $X = \{p \in V | f_i(p) = 0, i = 1, \dots, n\}$  has at most  $m_1 \cdots m_n$  elements.*

**Proof.** Let  $J$  be the radical of the ideal  $I$ . Then the nullstellensatz implies that the algebra of regular functions on  $X$  is  $\bar{R} = \mathcal{P}(V)/J$ . Let  $R = \mathcal{P}(V)/I$ . Then it is clear that  $\dim Gr^j \bar{R} \leq \dim Gr^j R = \dim(\mathcal{P}(V)/I_{top})^j$ . Now,  $I_{top} \supset I_g$ . Thus  $\dim(\mathcal{P}(V)/I_{top})^j \leq \dim R_g^j = h_j$  (all notation is as above). Thus  $\dim \bar{R} \leq \sum h_j = h(1) = m_1 \cdots m_n$ . The nullstellensatz implies that the elements of  $\bar{R}$  separate the points of  $X$ . The proposition now follows.

### 3. Some observations about isolated elements varieties over $\mathbb{C}$

If  $f : S^k \rightarrow S^k$  is a continuous map then  $\deg f$  is defined to be the action of  $f$  on  $H^k(S^k, \mathbb{Z}) \cong \mathbb{Z}$ . Set  $\omega = \sum_{i=1}^{k+1} (-1)^{i+1} x_i dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx \wedge \dots \wedge dx_{k+1}$  and  $C_k = \int_{S^k} \omega$ . If  $f$  is smooth then

$$\deg f = C_k^{-1} \int_{S^k} f^* \omega.$$

For our purposes this definition of  $\deg f$  is sufficient although it is only obvious that this integral representation of  $\deg f$  yields a real number. What is obvious is that if  $f : [0, 1] \times S^k \rightarrow S^k$  is smooth and if  $f_0(x) = f(0, x)$ ,  $f_1(x) = f(1, x)$  then  $\deg f_0 = \deg f_1$ . This follows from Stokes theorem.

The following result is taken from [M2, Lemma B.1, p.111].

**Lemma 3.1.** *If  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial map and  $p \in \mathbb{C}^n$  is such that*

$$(1) f(p) = 0.$$

$$(2) \det Df(p) \neq 0 \text{ (i.e. } \det \left[ \frac{\partial f_i}{\partial x_j}(p) \right] \neq 0).$$

*Then there exists  $r > 0$  such that if  $0 < \|x\| \leq r$  then  $f(x + p) \neq 0$ . Let  $0 < s \leq r$  and set  $\Phi_s(x) = \frac{f(p+sx)}{\|f(p+sx)\|}$  for  $x \in S^{2n-1}$ . Then  $\deg \Phi_s = 1$  for  $0 < s \leq r$ .*

**Proof.** We first note that if  $0 < s < r$  and we set  $h(t, x) = \Phi_{s+t(r-s)}$  then  $h_0 = \Phi_s$  and  $h_1 = \Phi_r$ . Thus  $\deg \Phi_s = \deg \Phi_r$  for  $0 < r < s$ . Set  $A = Df(p)$  then  $f(p+x) = Ax + E(x)$  and if  $\|x\| \leq r$  then  $\|E(x)\| \leq C\|x\|^2$  with  $C > 0$  fixed. Since  $\det A \neq 0$  there exists  $s$  with  $0 < s < r$  such that if  $\|x\| = s$  then  $\|E(x)\| < \frac{1}{2}\|Ax\|$ . Set  $g(t, x) = \frac{sAx + tE(sx)}{\|sAx + tE(sx)\|}$  for  $x \in S^{2n-1}$ . Then  $g_1 = \Phi_s$  and  $g_0(x) = \frac{Ax}{\|Ax\|}$ . Thus  $\deg \Phi_s = \deg g_0$ . Finally  $GL(n, \mathbb{C})$  is connected thus there exists a smooth curve  $\sigma(t)$  in  $GL(n, \mathbb{C})$  such that  $\sigma(0) = I$  and  $\sigma(1) = A$ . Set  $u(t, x) = \frac{\sigma(t)x}{\|\sigma(t)x\|}$ . Then  $u_0(x) = x$ ,  $x \in S^{2n-1}$  and  $u_1 = g_0$ . The Lemma follows.

**Lemma 3.2.** *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and let  $p \in \mathbb{C}^n$ . Suppose that there exists  $r > 0$  such that  $f(x) \neq 0$  for  $\|x - p\| \leq r$ . If  $0 < s \leq r$  then define  $\Phi_s$  as in Lemma 3.1. Then  $\deg \Phi_s = 0$ .*

**Proof.** Set  $g(t, x) = \Phi_{tr}(x)$ . Then  $g_0(x) = \frac{f(p)}{\|f(p)\|}$  for  $x \in S^{2n-1}$  and  $g_1 = \Phi_r$ . Clearly  $\deg g_0 = 0$ .

Here is an immediate implication:

**Proposition 3.3.** *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a polynomial mapping. Assume that  $p \in \mathbb{C}^n$  and  $f(x) \neq 0$  for  $\|x - p\| = r$ . Let  $\Phi_r$  be defined as in Lemma 3.1. If  $\deg \Phi_r \neq 0$  then there exists  $x \in \mathbb{C}^n$  such that  $\|x - p\| < r$  with  $f(x) = 0$ .*

We now combine this with the observations in the previous section.

**Theorem 3.4.** *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a polynomial map. Let  $\deg f_i = m_i$ . Then there are at most  $m_1 m_2 \cdots m_n$  elements  $p \in \mathbb{C}^n$  such that  $f(p) = 0$  and  $\det Df(p) \neq 0$ .*

**Proof.** We will use standard multindex notation. That is, if  $I = (i_1, \dots, i_n)$ ,  $i_j \in \mathbb{N} = \{0, 1, 2, \dots\}$  then  $|I| = i_1 + \dots + i_n$  and  $x^I = x_1^{i_1} \cdots x_n^{i_n}$ . We may assume  $m_i > 0$  for all  $i$  (otherwise we are discussing the empty set). Let  $f(x) = \sum a_I x^I$  with  $a_I \in \mathbb{C}^n$ . Given  $m > 0$ ,  $m \in \mathbb{Z}$  there exists  $g^m = (g_1^m, \dots, g_n^m)$  such that  $g_i^m$  is a polynomial on  $\mathbb{C}^n$ ,  $(g^m)_{top} \in \Omega_{m_1, \dots, m_n}$  and  $g^m = \sum_I b_I^m x^I$  with  $\|a_I - b_I^m\| < \frac{1}{m}$ . Let  $p_1, \dots, p_s$  be distinct elements with  $f(p_i) = 0$  and  $\det Df(p_i) \neq 0$ . Then there exists  $r > 0$  such that the sets  $B_{p_i}(r) = \{x \in \mathbb{C}^n \mid \|x - p_i\| \leq r\}$  are disjoint and if  $f(p) = 0$  with  $p \in B_{p_i}(r)$  then  $p = p_i$ . If  $x \in \mathbb{C}^n$  then

$$\|f(x) - g^m(x)\| \leq \sum_I \|a_I - b_I^m\| |x|^I.$$

Hence there exists a constant  $C > 0$  such that if  $x \in \cup B_{p_i}(r)$  then  $\|f(x) - g^m(x)\| < \frac{C}{m}$ . In particular this implies that if  $m$  is sufficiently large and if  $\|x - p_i\| = r$  then  $\|f(x)\| > \frac{1}{2} \|f(x) - g^m(x)\|$ . Set

$$h(t, x) = \frac{f(p_i + rx) + t(g^m(p_i + rx) - f(p_i + rx))}{\|f(p_i + rx) + t(g^m(p_i + rx) - f(p_i + rx))\|}.$$

Then  $h_0(x) = \Phi_r(x)$  (for  $f$ ) and  $h_1(x) = \frac{g^m(p_i + rx)}{\|g^m(p_i + rx)\|}$  for  $\|x\| = 1$ . Thus  $\deg h_1 = 1$  by Lemma 3.1. Hence Proposition 3.4 implies that there exists  $q_i \in B_{p_i}(r)$  such that  $g^m(q_i) = 0$ . Since the  $B_{p_i}(r)$  are mutually disjoint, Proposition 2.3 implies that  $s \leq m_1 \cdots m_n$ . This completes the proof.

We note that Lefschetz has shown that if  $p$  is an isolated 0 of a polynomial map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and if  $r > 0$  is such  $f(x) \neq 0$  for  $0 < \|x - p\| \leq r$  then if  $\Phi_s$  is as in Lemma 3.1 then  $\deg \Phi_s \geq 1$  for  $0 < s \leq r$  (cf. [M2, p.114]). The proof of Theorem combined with this result implies the following refinement:

**Theorem 3.4'.** *Let  $f$  be as in Theorem 3.4. Then there are at most  $m_1 \cdots m_n$  isolated zeros of  $f$ .*

In section 7 we will give an algebraic proof of a sharpening of this result.

#### 4. On Milnor's Theorem 1

Let  $f$  be a polynomial of degree  $k$  with real coefficients in  $n$  variables. We set  $X = X(f) = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ .

**Lemma 4.0.** *If  $X$  is compact and  $n \geq 2$  then  $k$  is even.*

**Proof.** Assume that  $k$  is odd. We show that  $f$  has arbitrarily large zeros. Let  $f = g + h$  with  $g$  homogeneous of degree  $k$  and  $\deg h < k$ . Let  $b \in \mathbb{R}^n$

be such that  $g(b) \neq 0$ . If  $a \in \mathbb{R}^n$  and  $\langle a, b \rangle = 0$  then  $\varphi(t) = f(a + tb) = t^k g(b) + u(t)$  with  $\deg u < k$ . Thus  $\varphi$  is a polynomial of degree  $k$  in  $t$ . Since  $k$  is odd,  $\varphi$  must have a real zero,  $\xi$ . Now,  $\|a + \xi b\| \geq \|a\|$  and  $a$  is arbitrary subject to  $\langle a, b \rangle = 0$ , the lemma follows.

We now assume that  $X$  is compact non-empty and  $n \geq 2$  (so  $k$  is even). We also assume that if  $x \in X$  then  $df_x \neq 0$ . Thus  $X$  is a smooth manifold of dimension  $n - 1$ . If  $\omega \in S^{n-1}$  then we set  $h_\omega(x) = \langle x, \omega \rangle$  for  $x \in X$ . If  $p \in X$  is a critical point for  $h_\omega$  then  $\omega$  must be orthogonal to the tangent space of  $X$  at  $p$ . Thus  $\omega$  must be a multiple of  $N(p) = \left( \frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right)$ . If  $x \in \mathbb{R}^n, x \neq 0$ , then we write  $[x]$  for the corresponding one dimensional subspace with basis  $x$ . That is,  $[x] \in \mathbb{P}^{n-1}(\mathbb{R})$  (real projective space of dimension  $n - 1$ ). Set  $\pi(x) = [N(x)], x \in X$ . Then  $\pi$  is a smooth mapping from  $X$  to  $\mathbb{P}^{n-1}(\mathbb{R})$ .  $p \in X$  is a critical point of  $h_\omega$  if and only if  $\pi(p) = [\omega]$ . Since  $X$  is compact, this implies that  $\pi$  is surjective.

Sard's theorem implies that the set of critical values (i.e. the set of  $\pi(p)$  such that  $d\pi_p$  is not bijective) has dense complement in  $\mathbb{P}^{n-1}(\mathbb{R})$  (see also Lemma 6.4). We make an orthogonal change of variables and we assume that  $[e_n]$  is not a critical value. (Here  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$ ). Set  $g = (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}})$ . Then  $g$  is a polynomial map of  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

**Lemma 4.1.** *The assumptions are as above. Let  $h = h_{e_n}$ . Then the set of critical points of  $h$  is precisely the set of points  $p \in \mathbb{R}^n$  such that  $g(p) = 0$  and at such a  $p$ ,  $\det Dg(p) \neq 0$ . Finally, each of the critical points in  $X$  of  $h$  is non-degenerate.*

Before we prove this Lemma we indicate how Milnor uses it.

**Theorem 4.2.** *Assume that  $n > 1$ . Let  $f$  be a polynomial with real coefficients in  $n$  variables. Assume that*

1.  $X = \{x \in \mathbb{R}^n | f(x) = 0\}$  is compact.
2. If  $x \in X$  then  $df_x \neq 0$ .

Then the number of connected components of  $X$  is at most  $\frac{k(k-1)^{n-1}}{2}$ .

**Note.** We using the Morse inequalities ([M3,I.5]) one can see that in fact one has the sum of the Betti numbers of  $X$  is at most  $k(k-1)^{n-1}$ .

**Proof.** We may use the assumptions and notation in Lemma 4.1. Then in light of Lemma 4.1, Theorem 3.4 implies that  $h$  has at most  $k(k-1)^n$  critical points in  $X$ . On each connected component of  $X$ ,  $h$  must have a maximum and a minimum. Since,  $h$  has only a finite number of critical points there must be at least 2 in each connected component. The Morse inequalities imply the note above.

We will now prove the lemma. If  $p \in X$  is a critical point for  $h$  then  $\pi(p) = [e_n]$ . Thus  $g(p) = 0$  and  $\frac{\partial f}{\partial x_n}(p) \neq 0$ . Write  $p = (p', p_n)$  then the imp theorem implies that there exists a neighborhood,  $U$  of  $p'$  in  $\mathbb{R}^{n-1}$  and a smooth function  $\varphi$  on  $U$  with  $\varphi(p') = p_n$  and  $f(y, \varphi(y)) = 0$  for  $y \in U$ . If  $1 \leq i \leq n-1$  then

$$0 = \frac{\partial f}{\partial y_i}(y, \varphi(y)) = \frac{\partial f}{\partial x_i}(y, \varphi(y)) + \frac{\partial \varphi}{\partial y_i}(y) \frac{\partial f}{\partial x_n}(y, \varphi(y)). \quad (1)$$

Our assumption implies that  $p'$  is a critical point of  $\varphi$ . Thus if we differentiate the above equation relative to  $y_j$  with  $1 \leq j \leq n-1$  and evaluate at  $y = p'$  we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(p) = -\frac{\partial f}{\partial x_n}(p) \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(p'). \quad (2)$$

We now use the assumption that  $[e_n]$  is a regular value of  $\pi$ . If  $q \in X$  is close to  $p$  then  $\frac{\partial f}{\partial x_n}(q) \neq 0$ . Thus

$$\pi(q) = [u_1(q), \dots, u_{n-1}(q), 1]$$

with  $u_i(q) = \frac{\partial f}{\partial x_i}(q) / \frac{\partial f}{\partial x_n}(q)$ ,  $i = 1, \dots, n-1$ . We calculate  $\frac{\partial u_i}{\partial y_j}(y, \varphi(y))$  at  $y = p'$  using the chain the fact that  $\frac{\partial \varphi}{\partial y_i}(p') = 0$  (see (1) above)  $i = 1, \dots, n-1$  and find that if  $1 \leq i, j \leq n-1$  then

$$\frac{\partial u_i}{\partial y_j}(p', \varphi(p')) = \frac{\frac{\partial^2 f}{\partial x_i \partial x_j}(p)}{\frac{\partial f}{\partial x_n}(p)}.$$

The assumption that  $[e_n]$  is a regular value of  $\pi$  means that  $\det \left[ \frac{\partial u_i}{\partial y_j}(p', \varphi(p')) \right] \neq 0$ . This combined with (2) above implies that  $p$  is a non-degenerate critical point of  $h$ . Also since  $\pi(p) = [e_n]$   $\det \left[ \frac{\partial g_i}{\partial x_j}(p) \right]_{1 \leq i, j \leq n} = \frac{\partial f}{\partial x_n}(p) \det \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right]_{1 \leq i, j \leq n-1}$ . The lemma now follows.

## 5. On Milnor's Theorem 2

We now show how Milnor derives his main theorem from Theorem 4.2. Set  $B(r) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ . Let  $f_1, \dots, f_m$  be polynomials with real coefficients of degree at most  $k$ . Following Milnor we set for  $\epsilon > 0$

$$u_\epsilon(x) = f_1(x)^2 + \dots + f_m(x)^2 + \epsilon^2 \|x\|^2$$



for  $\epsilon > 0$ . Set  $K(\epsilon, \delta) = \{x \in \mathbb{R}^n | u_\epsilon(x) \leq \delta^2\}$  for  $\delta > 0$  then  $K(\epsilon, \delta) \subset B(\frac{\delta}{\epsilon})$  hence compact. Set  $X = \{x \in \mathbb{R}^n | f_i(x) = 0, i = 1, \dots, m\}$ . If  $r \leq \frac{\delta}{\epsilon}$  then  $X \cap B(r) \subset K(\epsilon, \delta)$ . We assume that  $X$  is non-empty. Hence there exists  $r_0 > 0$  such that if  $r \geq r_0$  then  $B(r) \cap X \neq \emptyset$ . Fix  $r \geq r_0$ . Sard's theorem (cf. also Lemma 6.2) implies that there exists a sequence  $\{\epsilon_i\}$  with  $\epsilon_i \geq \epsilon_{i+1} > 0$  and  $\delta_i > 0$  such that  $\frac{\delta_i}{\epsilon_i} \geq \frac{\delta_{i+1}}{\epsilon_{i+1}}$ ,  $\lim_{i \rightarrow \infty} \frac{\delta_i}{\epsilon_i} = r$  and  $\delta_i$  is a regular value of  $u_{\epsilon_i}$ .

We note

**Lemma 5.1.**  $K(\epsilon_i, \delta_i) \supset K(\epsilon_{i+1}, \delta_{i+1})$  and  $\cap K(\epsilon_i, \delta_i) = X \cap B(r)$ .

**Proof.** We note that  $\frac{\delta_{i+1}}{\epsilon_{i+1}} \leq \frac{\delta_i}{\epsilon_i}$  implies that  $\delta_{i+1} \leq \frac{\delta_i \epsilon_{i+1}}{\epsilon_i} \leq \delta_i$  since  $\epsilon_i \geq \epsilon_{i+1}$ . Thus writing the inequality  $u_{\epsilon_{i+1}}(x) \leq \delta_{i+1}$  as

$$\frac{f_1(x)^2 + \dots + f_m(x)^2}{\delta_{i+1}} + \frac{\epsilon_{i+1} \|x\|^2}{\delta_{i+1}} \leq 1.$$

The asserted inclusions now follow from

$$\frac{f_1(x)^2 + \dots + f_m(x)^2}{\delta_i} + \frac{\epsilon_i \|x\|^2}{\delta_i} \leq \frac{f_1(x)^2 + \dots + f_m(x)^2}{\delta_{i+1}} + \frac{\epsilon_{i+1} \|x\|^2}{\delta_{i+1}}.$$

The above form of the definition of  $K(\epsilon_i, \delta_i)$  also implies the assertion about the intersection.

Set  $\partial K(\epsilon, \delta) = \{x \in \mathbb{R}^n | u_\epsilon(x) = \delta\}$ . For each  $i$ ,  $\partial K(\epsilon_i, \delta_i)$  and  $u_{\epsilon_i} - \delta_i$  satisfy (1) and (2) of Theorem 4.2. Thus we have

I. For each  $i$ , the number of connected components of  $\partial K(\epsilon_i, \delta_i)$  is at most  $k(2k - 1)^{n-1}$ .

**Lemma 5.2.** Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous map. If  $r \in \mathbb{R}$  then the number of connected components of  $u^{-1}((-\infty, r])$  is less than or equal to the number of connected components of  $u^{-1}(r)$ .

**Proof.** We may assume that  $Y \neq \emptyset$ . Let  $Y = u^{-1}((-\infty, r])$ ,  $Z = u^{-1}(r)$  then  $Y - Z$  is open in  $\mathbb{R}^n$  ( $Y - Z = \{x \in \mathbb{R}^n | u(x) < r\}$ ). Suppose that  $W$  is a connected component of  $Y$  and  $W \cap Z = \emptyset$ . There exists an open subset  $U$  of  $\mathbb{R}^n$  such that  $U \cap Y = W$ . Thus  $W = U \cap (Y - Z)$  which is open in  $\mathbb{R}^n$ . Hence  $W$  is open and closed in  $\mathbb{R}^n$  hence empty. This is a contradiction. Thus every connected component of  $Y$  has a non-empty intersection with  $Z$ . This implies the Lemma.

**Note.** In the case when  $u = u_{\epsilon_i}$  and  $r = \delta_i$ , Milnor shows (using Alexander duality) the sum of the Betti numbers of  $K(\epsilon_i, \delta_i)$  is less than or equal to half the sum of the Betti numbers of  $K(\epsilon_i, \delta_i)$ .

If  $Z$  is a closed subset of  $\mathbb{R}^n$  set  $b_0(Z)$  equal to the number of connected components of  $Z$ . In light of I Lemma 5.2 implies that

$$\text{II. } b_0(K(\epsilon_i, \delta_i)) \leq k(2k - 1)^{n-1}.$$

**Lemma 5.3.** *Let  $C_i$  be compact subsets of  $\mathbb{R}^n$  with  $C_i \supset C_{i+1}$  and  $\bigcap C_i = C$ . If  $b_0(C_i) \leq d$  then  $b_0(C) \leq d$ .*

**Proof.** We may assume that  $C \neq \emptyset$ . Let  $C = Y_1 \cup Y_2 \cup \dots \cup Y_s$  be the decomposition of  $C$  into connected components. Let  $C_{i,j}$ ,  $j = 1, \dots, d_i \leq d$  be the connected components of  $C_i$ . Then each  $Y_j$  is contained in a unique connected component  $C_{i,l(i,j)}$  of  $C_i$ . We note that  $C_{i,l(i,j)} \supset C_{i+1,l(i+1,j)}$ . We assert that for each  $j$  there exists  $r(j)$  such that if  $i \geq r(j)$  then  $C_{i,l(i,j)} \cap C = Y_j$ . This will clearly prove the lemma. Suppose not then there is an infinite sequence  $r_1 < r_2 < \dots$  and  $z_k \in C_{r_k,l(r_k,j)} - Y_j$ ,  $z_k \in Y$ . Taking a subsequence, if necessary, we may assume  $\lim_{k \rightarrow \infty} z_k = z_0$  and  $z_0 \in Y$ . But  $z_0 \in \bigcap C_{r_k,l(r_k,j)} = V_j$  which is connected. Since  $V_j \supset Y_j$ ,  $V_j = Y_j$  and we have a contradiction.

This lemma combined with II and Lemma 5.1 implies

$$\text{III. } b_0(B(r) \cap X) \leq k(2k - 1)^{n-1}.$$

**Note.** Using standard properties of Čech cohomology (commuting with infinite decreasing sequences of compact spaces) Milnor has the same estimate as in III for the sum of the Betti numbers.

We can now prove the main theorem.

**Theorem 5.4.** *Let  $f_1, \dots, f_m$  be polynomials in  $n$  variables with real coefficients. Let  $X = \{x \in \mathbb{R}^n \mid f_i(x) = 0, i = 1, \dots, m\}$ . Then the number of connected components of  $X$  is less than or equal to  $k(2k - 1)^{n-1}$ .*

**Proof.** We may assume that  $X \neq \emptyset$ . Let  $X_1, \dots, X_s$  be connected components of  $X$ . Let  $r > 0$  be so large that  $B(r) \cap X_i \neq \emptyset$  for  $i = 1, \dots, s$ . Then  $B(r) \cap X_i$  is a union of  $p_i > 0$  connected components. Since  $\bigcup_i B(r) \cap X_i$  is open and closed in  $B(r) \cap X$  this implies  $s \leq p_1 + \dots + p_s \leq b_0(B(r) \cap X) \leq k(2k - 1)^{n-1}$ . The theorem follows.

**Note.** Milnor in fact gives the same inequality for the sum of the Betti numbers. Here he uses a more sophisticated argument (in fact two). We recommend that the reader consult the original paper of Milnor. We also note that [T, Lemme 3., p.260] also implies this assertion.

We also observe that the argument above implies that if  $f_p(x) = \prod_{j=1}^k (x_p - j)$ ,  $p = 1, \dots, n$ , if  $r > k$  and (in the notation above) if  $i$  is sufficiently large then  $\partial K(\epsilon_i, r_i)$  is a union of at least  $k^n$  connected smooth manifolds.

In [M1], Milnor notes that he can find no examples with the sum of the Betti numbers greater than  $k^n$ . This suggests the following

**Problem.** Can we replace  $k(2k - 1)^{n-1}$  in Theorem 5.4 with  $k^n$ ?

In section 8 we will give an affirmative answer to this question in the case when  $n = 2$ . In section 9 we will give an affirmative answer for the sum of the Betti numbers for a non-singular hypersurface.

## 6. Some further results

Let  $f_1, \dots, f_m, g_1, \dots, g_q$  be polynomials in  $n$  variables with real coefficients. Let  $X = \{x \in \mathbb{R}^n | f_i(x) = 0, i = 1, \dots, m\}$ ,  $Y = \{x \in \mathbb{R}^n | g_j(x) = 0, j = 1, \dots, q\}$ . We will now show how the result of Milnor and Thom applies to  $X - Y = \{x \in X | x \notin Y\}$ . We first observe that if  $x \notin Y$  then some  $g_i(x) \neq 0$ . This since the polynomials  $g_i$  have real coefficients this is the same as  $h(x) = \sum g_j(x)^2 \neq 0$  (if  $m = 1$  set  $h = g_1$ ). Thus  $X - Y$  is homeomorphic with

$$\{(x, t) \in \mathbb{R}^{n+1} | f_i(x) = 0, i = 1, \dots, m, th(x) = 1\}.$$

We can now apply the result to the variety given in this way.

**Theorem 6.1.** *Let  $k = \max_{i,j} \{\deg f_i, 2 \deg g_j + 1\}$  if  $q > 1$  and  $k = \max\{\deg g_1 + 1, \deg f_i\}$  if  $q = 1$ . Then the sum of the Betti numbers of  $X - Y$  is less than or equal to  $k(2k - 1)^n$ .*

This in particular applies to the situation  $\mathbb{R}^n - Y$ . An important example of this is the case  $g = \prod_{i < j} (x_i - x_j)$ . Then  $\deg g = \frac{n(n-1)}{2}$ . Let  $Y = \{x \in \mathbb{R}^n | g(x) = 0\}$ . Then it standard that  $\mathbb{R}^n - Y$  is a union of  $n!$  non-empty convex subsets of  $\mathbb{R}^n$ . The estimate of Theorem 6.1 is

$$\left(\frac{n(n-1)}{2} + 1\right)(n(n-1) + 1)^n \sim n^{2n+2}/2.$$

We will come back to this example in section 9.

We also record algebraic variants of Sard's theorem that are used in the proofs of Milnor's theorems. Let  $f$  be a polynomial in  $n$  variables with complex coefficients (for this any field of characteristic 0 will do). Let  $\Sigma(f) = \{x \in \mathbb{C}^n | df_x = 0\}$ . The following results will use a bit more algebraic geometry than the rest of the exposition.

**Lemma 6.2.** *The set  $f(\Sigma(f))$  is finite.*

**Proof.** Let  $Y$  be an irreducible component of  $\Sigma(f)$ . We show that  $f(Y)$  is a point in  $\mathbb{C}$ . Since  $Y$  is irreducible the Zariski closure of  $f(Y)$  is irreducible

as a subvariety of  $\mathbb{C}$ . Thus the Zariski closure of  $f(Y)$  is either a point or all of  $\mathbb{C}$ . Suppose that we are in the latter situation. Let  $Y^\circ$  be the set of simple points of  $Y$ . Then  $Y^\circ$  is a non-singular quasi-affine variety and the Zariski closure of  $Y^\circ$  is  $Y$ . Thus  $f(Y^\circ)$  has Zariski interior in  $\mathbb{C}$ . But  $f$  is constant on each connected component of  $Y^\circ$  in the classical topology of  $Y^\circ$  (the subspace topology of  $Y^\circ$  in  $\mathbb{C}^n$  with the Euclidian metric topology). Since  $Y^\circ$  has only a countable number of connected components, we have a contradiction.

A similar elementary argument using algebraic geometry proves the following result.

**Lemma 6.3.** *Let  $X$  be an irreducible smooth  $n$ -dimensional affine variety over  $\mathbb{C}$  and let  $f : X \rightarrow Y$  where  $Y = \mathbb{C}^n$  or  $Y = \mathbb{P}^n$ . Let  $\Sigma = \{x \in X \mid df_x \text{ is not surjective}\}$  then the closure of  $f(\Sigma)$  in the Zariski topology of  $Y$  has dimension at most  $n - 1$ .*

**Corollary 6.4.** *Let  $\varphi$  be a polynomial in  $n$  indeterminates with real coefficients and let  $X = \{x \in \mathbb{C}^n \mid \varphi(x) = 0, d\varphi_x \neq 0\}$ . Let  $f : X \rightarrow \mathbb{P}^{n-1}$  be a regular map such that  $f(X \cap \mathbb{R}^n) \subset \mathbb{P}^{n-1}(\mathbb{R})$ . Let  $\Sigma = \{x \in X \mid df_x \text{ is not surjective}\}$  then there exists a non-zero, homogeneous polynomial,  $u$ , with real coefficients such that  $f(\Sigma \cap \mathbb{R}^n) \subset \{x \in \mathbb{P}^{n-1}(\mathbb{R}) \mid u(x) = 0\}$ .*

## 7. An estimate on the number of irreducible components

In this section  $K$  will denote an algebraically closed field. We will be using a bit more algebraic geometry that was needed in the earlier sections. Let  $\mathbb{P}^n$  denote the  $n$  dimensional projective space over  $K$  and  $\mathbb{A}^n$  the  $n$  dimensional affine space. This section will be devoted to the proof of the following result.

**Theorem 7.1.** *Let  $X$  be (Zariski) closed in  $\mathbb{A}^n$  (resp.  $\mathbb{P}^n$ ) given as the zero locus of polynomials  $f_1, \dots, f_m$  (resp. homogeneous) of degree at most  $k$ . Then the number of irreducible components of  $X$  is at most  $k^n$ .*

We first note that the affine case follows from the projective case. Indeed, by adding a variable  $x_0$  we can homogenize  $f_1, \dots, f_m$  to be homogeneous of degree  $k$ . Let  $Y$  be the corresponding projective variety. Let  $Y = \cup_{i=1}^d Y_i$  be an irredundent decomposition into irreducible components. Then  $X = \cup_{i=1}^d Y_i \cap \mathbb{A}^n$ . Now throw away the redundant terms (using the fact that  $\mathbb{A}^n \cap Y_i$  is open in  $Y_i$  and hence irreducible as an affine variety).

We now concentrate on the projective case. We may assume that all of the  $f_i$  are homogeneous of degree  $k$  in variables  $x_0, x_1, \dots, x_n$ . If  $X \subset \mathbb{P}^n$  is

closed and irreducible then  $\deg X$  is the leading coefficient of  $(\dim X)!h(t)$  with  $h$  the Hilbert polynomial of the homogeneous coordinate ring of  $X$ . In the projective case we will prove the following sharper result

**Theorem 7.1'.** *Let  $f_1, \dots, f_m$  be homogeneous polynomials of degree  $k$  and let  $X$  be the zero locus of  $\{f_1, \dots, f_m\}$  in  $\mathbb{P}^n$ . Let  $X = X_1 \cup \dots \cup X_d$  be an irredundent decomposition of  $X$  into irreducible components then*

$$\sum_i \deg X_i \leq k^n.$$

The following simple lemma will be used in the proof of the theorem.

**Lemma 7.2.** *Let  $X \subset \mathbb{P}^n$  be closed. Let  $V$  be a subspace of  $K[x_0, \dots, x_n]$  consisting of homogeneous elements of degree  $k$ . Then we can label the irreducible components of  $X$  as  $X_1, \dots, X_d$  with  $V|_{X_i} = 0$  for  $i \leq s$  and there exists  $f \in V$  such that  $f|_{X_i} \neq 0$  for  $i > s$ .*

**Proof.** Order the index set  $\{1, \dots, d\}$  by inclusion. Let  $S$  be a maximal subset subject to the condition  $V|_{X_i} = 0$  for  $i \in S$ . Set  $S^c = \{i \notin S | 1 \leq i \leq d\}$ . Assume that for each  $f \in V$ ,  $f \neq 0$ , there exists  $i \in S^c$  such that  $f|_{X_i} \neq 0$ . Let for  $i \in S^c$ ,  $V_i = \{f \in V | f|_{X_i} = 0\}$ . Then  $V = \bigcup_{i \in S^c} V_i$ . This implies that there exists  $i \in S^c$  such that  $V|_{X_i} = 0$ . This contradicts the maximality of  $S$ . Thus there exists  $f \in V$  such that  $f|_{X_i} \neq 0$  for  $i \in S^c$ . This completes the proof of the Lemma.

We now prove Theorem 7.1'. Obviously, we may assume  $n \geq 2$ . We prove the result by induction on  $k$ . If  $k = 1$  then the result is obvious. Assume the result for  $1, \dots, k - 1$ . Let  $\mathcal{P}^k$  denote the space of  $f \in K[x_0, \dots, x_n]$  that are homogeneous of degree  $k$ . Set  $V = \{f \in \mathcal{P}^k | f|_X = 0\}$ . Then  $X = \{x \in \mathbb{P}^n | V(x) = 0\}$ . Choose  $g_1 \in V$ ,  $g_1 \neq 0$ . Let  $Y_1 = \{x \in \mathbb{P}^n | g_1(x) = 0\}$ . Apply Lemma 7.2 to  $Y_1$  and find that  $Y_1 = X_1 \cup X_2 \cup \dots \cup X_{s_1} \cup Z_1 \cup \dots \cup Z_{t_1}$  an irredundent decomposition into irreducible components such that  $V|_{X_i} = 0$  for  $i = 1, \dots, s_1$  and if  $t_1 > 0$  then there exists  $g_2 \in V$  such that  $g_2(Z_i) \neq 0$  for  $i = 1, \dots, t_1$ . If  $t_1 = 0$  then  $\{X_1, \dots, X_{s_1}\}$  is the set of irreducible components of  $X$ . Since  $g$  has degree  $k$ ,  $\sum_{i \leq s_1} \deg X_i \leq k$  and the result is proved in this case. Now assume that  $t_1 > 0$ . If  $s_1 > 0$  then there must be a non-trivial irreducible factor,  $h$ , of  $g_1$  that divides every element of  $V$ . Then  $X = X(h) \cup X(V/h)$  (here if  $S$  is a set of homogenous polynomials then  $X(S) = \{x \in \mathbb{P}^n | f(x) = 0, x \in S\}$ ). The inductive hypothesis applies to  $X(h)$  and to  $X(V/h)$ . If  $a > 0$  and  $b > 0$  that  $a^n + b^n < (a + b)^n$ . So the result follows if  $s_1 > 0$ . Thus we may assume that  $X = \bigcup Z_i$ . Thus  $Y_2 = \{z \in Y_1 | g_2(z) = 0\}$ . Let

$Y_2 = X_1 \cup \cdots \cup X_{s_2}$  be an irredundent decomposition into irreducible components. We note that Bezout's theorem (cf. [H; Theorem I.7.7, p. 53]) implies that there are integers  $c_i > 0$  such that

$$\sum c_i \deg(X_i) \leq \deg g_1 \deg g_2 = k^2.$$

Now  $\dim X_i = n - 2$ . Thus if  $n = 2$  then  $Y_2$  is a finite set with at most  $k^2$  elements. Since  $X \subset Y_2$ , the Theorem 7.1 is now completely proved for  $n = 2$ . We thus assume  $n > 2$ . If  $Y_2 = X$  we are also done. Otherwise, we can write  $Y_2 = X_1 \cup X_2 \cup \cdots \cup X_{p_2} \cup Z_1 \cup \cdots \cup Z_{q_2}$  (irreducible decomposition into irreducibles) with  $V|_{X_i} = 0$  and there exists  $g_3 \in V$  with  $g_3(Z_i) \neq 0$ . Set

$$Y_3 = X_1 \cup \cdots \cup X_{p_2} \cup \cup_i Z_i \cap X(g_3) \\ \{z \in \mathbb{P}^n | g_i(z) = 0, i = 1, 2, 3\}.$$

Let  $Z_i \cap X(g_3) = \cup_{j=1}^{q_{ij}} X_{ij}$  be an irredundent decomposition into irreducible components. Then applying Bezout's theorem there exist  $c_{ij} > 0$ ,  $c_{ij}$  integers such that

$$\sum_j c_{ij} \deg X_{ij} \leq \deg Z_i \deg g_3.$$

Since  $k > 1$  we see that we can write

$$Y_3 = X_1 \cup X_2 \cup \cdots \cup X_{p_2} \cup X_{p_2+1} \cup \cdots \cup X_{p_3}$$

with  $X_i$  irreducible and such that there exist positive integers  $d_{2i}$  so that

$$\sum_i d_{2i} \deg X_i \leq k^3.$$

Also,  $\dim X_i = n - 3$  for  $i > p_2$ . If  $n = 3$  we can argue as in the case of  $n = 2$  to complete the proof of the theorem. So assume  $n > 3$ . Then either  $Y_3 = X$  or we can continue the argument to find  $g_4 \in V$ . Obviously, we can continue this process. If the process continues to  $n$  stages then we are done as in the case of  $n = 2$ . If it stops in  $r < n$  steps then we have an upper bound on the sum of the degrees of the irreducible components of the form  $k^r$ . The theorem now follows.

## 8. A sharp result in the case $n = 2$

The purpose of this section is to prove

**Theorem 8.1.**  $f_1, \dots, f_m$  be polynomials with real coefficients in 2 variables of degree at most  $k$ . Then  $X = \{x \in \mathbb{R}^2 \mid f_i(x) = 0, i = 1, 2\}$  has at most  $k^2$  connected components.

We will prove this result by induction on  $k$ . If  $k = 0, 1$  then the result is obvious. So assume the theorem for degrees at most  $k - 1$ . Let  $V$  denote the span over  $\mathbb{C}$  of  $f_1, \dots, f_m$ . If  $S$  is a set of polynomials (with real or complex coefficients) set  $X_{\mathbb{R}}(S) = \{x \in \mathbb{R}^2 \mid f(x) = 0, f \in S\}$ . Let  $g \in V$  be an element of minimal degree. If for each  $h \in V$  there exists a non-constant irreducible factor,  $u$ , of  $g$  such that  $u$  divides  $h$ . Then there exists a non-constant factor,  $u$ , of  $g$  that divides every element of  $V$ . Then  $X = X_{\mathbb{R}}(u) \cup X_{\mathbb{R}}(V/u)$ . Thus if  $\deg u = r < k$  then  $b_0(X) \leq b_0(X_{\mathbb{R}}(u)) \cup b_0(X_{\mathbb{R}}(V/u)) \leq r^2 + (k - r)^2$ , by the inductive hypothesis. So we may assume that  $\deg u = k$ . But then  $u = g$  and hence  $X = X_{\mathbb{R}}(g)$ . Suppose that there exists  $h$  in  $V$  such that  $h$  and  $g$  are relatively prime. Then (notation as in section 7).  $\dim X(g, h) = 0$  and  $X \subset X(g, h)$ .  $X(g, h)$  has at most  $k^2$  elements by Theorem 7.1. Thus we are left with the case when  $X = X_{\mathbb{R}}(g)$  with  $g$  irreducible over  $\mathbb{C}$ . Assume that  $g$  is not a multiple of a polynomial with real coefficients. Let  $g(x) = u(x) + iv(x)$  with  $u, v$  polynomials with real coefficients. If  $u$  and  $v$  have a non-trivial (complex) factor  $w$  in common then  $w$  divides  $g$  which is contrary to our assumption. Thus  $u$  and  $v$  are relatively prime over  $\mathbb{C}$ . Since  $X = X_{\mathbb{R}}(u, v) \subset X(u, v)$  which is fini Theorem 7.1 implies the result in this case. We are thus left with the case when  $g$  has real coefficients and is irreducible over  $\mathbb{C}$ .

If  $a, b \in \mathbb{R}$  then set

$$u_{a,b}(x, y) = (x - a) \frac{\partial f}{\partial y}(x, y) - (y - b) \frac{\partial f}{\partial x}(x, y).$$

Suppose that for every  $(a, b) \in \mathbb{R}^2 - X$ ,  $g$  divides  $u_{a,b}$ . Then  $u_{a,b}(X(g)) = 0$  for all  $a, b \in \mathbb{R}^2$ . Differentiating this identity implies that  $X(g)$  has no simple points. We may thus choose  $(a, b) \in \mathbb{R}^2 - X$  such that  $g$  and  $u_{a,b}$  are relatively prime over  $\mathbb{C}$ . Fix such a  $u = u_{a,b}$ . Let  $X_1, \dots, X_l$  be the connected components of  $X$ . We assume that if  $1 \leq i \leq q$  and if  $p \in X_i$  then at least one of  $\frac{\partial f}{\partial x}(p)$ ,  $\frac{\partial f}{\partial y}(p)$  is non-zero. We also assume that if  $i > q$  then  $X_i$  contains a point where both of the partials are equal to 0. Set  $\varphi(x, y) = \frac{1}{2}((x - a)^2 + (y - b)^2)$ . Since each  $X_i$  is closed,  $\varphi$  must attain a minimum in  $X_i$ . If  $1 \leq i \leq q$  then at such a minimum,  $p$ ,  $0 = d\varphi_p \wedge df_p = u(p)dx \wedge dy$ . We therefore see that for each  $1 \leq i \leq l$  there exists an element  $p \in X_i$  such that  $p \in X(g, u)$ . But  $X(g, u)$  has at most  $k^2$  elements by Theorem 7.1.

### 9. A better estimate for smooth algebraic hypersurfaces

Let  $f$  be a polynomial with real coefficients in  $n$  indeterminates. Set  $X = \{x \in \mathbb{R}^n | f(x) = 0\}$ . The purpose of this section is to prove

**Theorem 9.1.** *If for every  $x \in X$ ,  $df_x \neq 0$  then  $b_0(X) \leq (\deg f)^n$ .*

We note that if  $k = \deg f$  then the Milnor-Thom theorem would give the inequality  $k(2k - 1)^{n-1}$ .

The proof of Theorem 9.1 is based on the following simple result.

**Lemma 9.2.** *Let  $V$  be a finite dimensional subspace of the polynomials in  $n$  variables over  $\mathbb{C}$ . Let  $X(V) = \{x \in \mathbb{C}^n | V(x) = 0\}$ . If  $x \in X(V)$  and if  $\dim\{dg_x | g \in V\} = n$  then there is a Zariski open subset  $U$  of  $\mathbb{C}^n$  such that  $U \cap X(V) = \{x\}$ .*

**Proof.** Let  $f_1, \dots, f_n \in V$  be such that

$$df_1 \wedge \dots \wedge df_n = \varphi dx_1 \wedge \dots \wedge dx_n$$

is non-zero at  $x$ . So  $\varphi(x) \neq 0$ . Let  $U_1 = \{y \in \mathbb{C}^n | \varphi(y) \neq 0\}$ . Then  $U \cap X(f_1, \dots, f_n)$  is isomorphic with the variety  $Y = \{(y, t) | f_i(y) = 0, i = 1, \dots, n, \varphi(y)t = 1\}$ . Set  $u_i(y, t) = f_i(y)$ ,  $i = 1, \dots, n$  and  $u_{n+1}(y, t) = \varphi(y)t - 1$ .

$$du_1 \wedge \dots \wedge du_{n+1} = \varphi^2 dy_1 \wedge \dots \wedge dy_n \wedge dt.$$

This implies that  $\dim Y = 0$ . So  $Y$  is finite. Set  $Y - \{x\} = F$  and  $U = U_1 - F$ . Then

$$\{x\} \subset U \cap X(V) \subset U \cap X(f_1, \dots, f_n) = \{x\}.$$

This completes the proof of the lemma.

We now prove Theorem 9.1. Clearly, we may assume that  $n \geq 2$ . Let  $a \in \mathbb{R}^n$  be such that if  $\alpha(x) = \frac{1}{2} \sum_i (x_i - a_i)^2$  then  $\alpha$  has non-degenerate critical points in  $X$  (such an  $a$  exists by the Lemma of Andrioti-Frankel (cf. [M3, Theorem 6.6, p.36]). If  $1 \leq i < j \leq n$  we set  $\psi_{ij}(x) = (x_i - a_i) \frac{\partial f}{\partial x_j}(x) - (x_j - a_j) \frac{\partial f}{\partial x_i}(x)$ . Then

$$d\alpha \wedge df = \sum_{i < j} \psi_{ij} dx_i \wedge dx_j.$$

Thus  $y \in X$  is a critical point for  $\alpha$  if and only if  $f(x) = \psi_{ij}(x) = 0$  for  $1 \leq i, j \leq n$ . Thus if  $V$  is the complex span of  $\{f\} \cup \{\psi_{ij} | i < j\}$ . Then the set of critical points of  $\alpha$  is  $X(V) \cap \mathbb{R}^n$ .



Let  $p \in X$  be a critical point for  $\alpha$ . After relabeling coordinates we may assume  $\frac{\partial f}{\partial x_n}(p) \neq 0$ . Set  $p = (p', p_n)$ . We can find a neighborhood  $U$  of  $p'$  in  $\mathbb{R}^{n-1}$  and a smooth function  $\varphi : U \rightarrow \mathbb{R}$  such that  $\varphi(p') = p_n$  and  $f(y, \varphi(y)) = 0$  for all  $y \in U$ . The condition that  $p$  is a non-degenerate critical point is just that

$$\Delta(p') = \det \left[ \frac{\partial^2 \alpha}{\partial y_i \partial y_j}(y, \varphi(y)) \right]_{y=p'} \neq 0.$$

Assume that  $i \leq n-1$  then

$$\frac{\partial \alpha}{\partial y_i}(y, \varphi(y)) = (y_i - a_i) + (\varphi(y) - a_n) \frac{\partial \varphi}{\partial y_i}(y).$$

Also

$$\frac{\partial \varphi}{\partial y_i}(y) = - \left( \frac{\partial f}{\partial x_n}(y, \varphi(y)) \right)^{-1} \frac{\partial f}{\partial x_i}(y, \varphi(y)).$$

So it follows that

$$\frac{\partial \alpha}{\partial y_i}(y, \varphi(y)) = \left( \frac{\partial f}{\partial x_n}(y, \varphi(y)) \right)^{-1} \psi_{i,n}(y, \varphi(y)).$$

Set  $u_i = \psi_{i,n}$ ,  $i = 1, \dots, n-1$  and  $u_n = f$ . Then a direct calculation (similar to the one in section 4) shows that

$$\det \left[ \frac{\partial u_i}{\partial x_j}(p) \right] = \left( \frac{\partial f}{\partial x_n}(p) \right)^{n+1} \Delta(p').$$

We can now apply Lemma 9.2 to conclude that each critical point of  $\alpha$  in  $X$  is an irreducible component of  $X(V)$ . Theorem 7.1 now implies that  $\alpha$  has at most  $k^n$  ( $k = \deg f$ ) critical points.

Since each connected component of  $X$  is closed  $\alpha$  must have a minimum on each connected component. Thus each connected component contains at least one critical point. This completes the proof of Theorem 9.1.

We note that the function  $\alpha$  in the proof above is proper. We may thus apply [M3, Theorem 3.5,p.20] to deduce

**Theorem 9.1'.** *Let  $f, X$  be as in Theorem 9.1. Then the sum of the Betti numbers of  $X$  is at most  $k^n$  ( $k = \deg f$ ).*

We note that if  $f$  is a non-zero polynomial in  $n$  indeterminates with real coefficients then  $U = \{x \in \mathbb{R}^n | f(x) \neq 0\}$  is isomorphic with the smooth

hypersurface  $Y = \{(x, t) | f(x)t = 1\}$  in  $\mathbb{R}^{n+1}$ . Thus Theorem 9.1' applies and we have

**Corollary 9.2.** *Let  $f$  and  $U$  be as above and assume that  $\deg f = k$ . Then the sum of the Betti numbers of  $U$  is at most  $(k + 1)^{n+1}$ .*

In the example of section 6, this theorem improves the estimate by a factor of  $2^{-n}$ .

### References

- [B] M. Ben-Or, Lower bounds for algebraic computation trees, in *Proc. 15th ACM Symposium on the Theory of Computing*, May 1983, 80-86.
- [H] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977.
- [M1] J. Milnor, On the Betti numbers of real varieties, *Proc. Amer. Math. Soc.*, **15** (1964), 275-280.
- [M2] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Math Studies, Study 61, Princeton, 1968.
- [M3] J. Milnor, *Morse Theory*, Annals of Math Studies, Study 51, 1963.
- [T] R. Thom, Sur l'homologie des variétés algébriques réelles, in *Differential and Combinatorial Topology*, Princeton University Press, 1965, 255-265.
- [W] H. Whitney, Elementary structure of real algebraic varieties, *Ann. of Math.*, **66** (1957), 545-546.

Department of Mathematics  
University of California at San Diego  
La Jolla, CA 92093

Received June 1995