Practice Problems for 104a Final Exam

1. Calculate the GCD of 3822 and 4185.

2. Find all integral solutions of the following equations in unknowns $x, y$.
   a) $21x + 51y = 27$.
   b) $12x + 20y = 5$.

3. Let $a, b \in \mathbb{Z}$ be such that $\gcd(a, b) = 1$. Show that if $ab = m^3$ with $m \in \mathbb{Z}$ then there exist $x, y \in \mathbb{Z}$ so that $a = x^3$ and $b = y^3$.

4. Prove that if $a, b \in \mathbb{Z}$ and $a^3 | b^2$ then $a | b$.

5. Show that if $N \in \mathbb{Z}$ and $N \geq 24$ then there exist $x, y \in \mathbb{N}$ so that $N = 5x + 7y$. Show that there are exactly 12 elements of $\mathbb{N}$ that cannot be written in the form $7x + 11y$ with $x, y \in \mathbb{N}$. (Hint: The following is a table of numbers $5x + 7y$ with the number placed in the $x^{th}$ row and the $y^{th}$ column with the labels starting with 0.)

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Observe that 24, 25, 26, 27, 28 are in the grid.)

6. Suppose that you have only pennies (1 cent), dimes (10 cents) and quarters (25 cents) and you want to buy something that costs $3.00. Prove that you can do this with 36 of your coins but not with 35.

7. For which primes $p$ does $x^2 + x + 1 = 0$ have a solution in $F_p$? If it has solution in $F_p$ how many does it have?

8. Find an integer that has a remainder of 9 when divided by 10 or 11 but is divisible by 13.

9. How many positive divisors does 14175 have? (Hint: It’s largest prime divisor is 7.)
10. Consider the field $F_{11}$ show that 2 is a primitive element (root) of $F_{11}$. Find all of the primitive elements of $F_{11}$.

11. Let $p$ be an odd prime and let $\left( \frac{a}{p} \right)$ be the Legendre symbol. Prove that

\[ \sum_{i=1}^{p-1} \left( \frac{i}{p} \right) = 0. \]

(Hint: Write this sum using a primitive element.)

12. Calculate the following Legendre symbols.
   a) $\left( \frac{229}{379} \right)$  
   b) $\left( \frac{7919}{12553} \right)$.