Math 140B. Foundations of Analysis

Lecture 1: The contraction mapping theorem.

There is a cool theorem that we did not have time to cover in 140A. Here is a problem which it can solve:

Problem. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x^3 + 1}{4}.$$

Set $x_0 = 1/2$, and define $x_1, x_2, \ldots$ by $x_j = f(x_{j-1})$. Then show that (x_n) converges to a point p with $f(p) = p$.

Definitions. Let (S, d) be a **metric space**. We say that the map $f : S \to S$ is a **contraction** of S if there exists $\alpha < 1$ such that

$$d(f(x), f(y)) \;\le\; \alpha\, d(x, y), \qquad \text{for all } x, y \in S.$$

We say that p is a **fixed point** of f if $f(p) = p$.

Quiz: Is a contraction continuous? Is it uniformly continuous?

Theorem (Contraction mapping theorem or Fixed point theorem for contractions). *If (S, d) is a* **complete** *metric space and $f : S \to S$ is a contraction, then f has a unique fixed point $p \in S$.*

In fact, if x is any point of S and we define $x_0 = x$, and $x_1, x_2, \ldots$ are defined by $x_j = f(x_{j-1})$, (so $x_j = f^j(x)$) then (x_j) converges to the unique fixed point p.

Proof of the contraction mapping theorem. Take any point $x \in S$ and define the sequence $x_j = f^j(x)$ as above. We will show that x_n is Cauchy. Indeed, we have

$$d(x_{n+1}, x_n) \;=\; d(f(x_n), f(x_{n-1}) \;\le\; \alpha\, d(x_n, x_{n-1}).$$

By induction, we have

$$d(x_{n+1}, d_n) \;\le\; \alpha^n d(x_1, x) \;=\; c\, \alpha^n, \qquad \text{where } c = d(x_1, x).$$

But then by the triangle inequality,

$$\begin{aligned} d(x_{n+k}, x_n) \;\le\; d(x_{n+k}, x_{n+k-1}) \;+\; \ldots \;+\; d(x_{n+1}, x_n) \;&\le\; (\alpha^{n+k-1} + \cdots + \alpha^n)c \\ &\le\; c \sum_{j=0}^{\infty} \alpha^{n+j} \;=\; \frac{c\alpha^n}{1 - \alpha}. \end{aligned}$$

But since α^n converges to zero, we see that the sequence (x_n) is Cauchy. Since S is complete this sequence converges to some limit p. Now

$$\lim_{n\to\infty} x_n = p, \qquad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} \;=\; p.$$

Since f is a contraction it is continuous and $f(p) = p$, so we have found a fixed point of f. If q is also a fixed point of f then $f(q) = q$ and

$$d(p,q) \;=\; d(f(p), f(q)) \;\leq\; \alpha\, d(p,q).$$

Hence $d(p,q) = 0$ and $p = q$. So f has a unique fixed point. Notice that the sequence (x_n) converges to this unique fixed point.

Solution of our problem. In order to show that (x_j) converges, we just need to show that f is a contraction of a complete metric space. Now $\mathbb{R}$ is a complete metric space, but is f a contraction of $\mathbb{R}$?

$$f(x) - f(y) \;=\; \frac{x^3 - y^3}{4} \;=\; \frac{x^2 + xy + y^2}{4}(x - y).$$

We want this bounded by

$$\alpha\, |x - y|$$

for some $\alpha < 1$. Thus we need

$$(*) \qquad \left|\frac{x^2 + xy + y^2}{4}\right| \;\leq\; \alpha \;<\; 1$$

on S. Now certainly this will not be true if $x > 2$ and $y > 0$. We must find a set $S \subset \mathbb{R}$ which contains $1/2$, and find a constant $\alpha < 1$ such that (*) holds on S. We can take $S = [0,1]$. On $[0,1]$ we have

$$|f(x) - f(y)| \;\leq\; \frac{3}{4}|x - y|.$$

In addition note that $[0,1]$ is complete (**why?**). We do need to check that $f : [0,1] \to [0,1]$. But this holds because f is increasing on $[0,\infty)$ and $f(0) = 1/4$, $f(1) = 1/2$. Hence $f[0,1] = [1/4, 1/2] \subset [0,1]$.

Hence by the contraction mapping theorem, (x_n) converges to the unique point $p \in [0,1]$ with $f(p) = p$.

Remarks 1. The contraction mapping theorem has many important uses in analysis. It is used to prove the existence of solutions to ordinary differential equations. It is used to prove the inverse function theorem, and it can be used to prove the convergence of numerical iteration schemes such as the Newton-Raphson method.
2. Soon we will have a good way to check whether functions $f : [a,b] \to \mathbb{R}$ are contractions: The **mean value theorem**. This states that if f is differentiable on $[a,b]$ then $f(x) - f(y) = f'(c)(x-y)$ for some c between x and y. For example, in our previous example with $f(x) = (x^3+1)/4$, we have $f'(c) = 3c^2/4$. We can therefore see that if $|f'(c)| < 1$ we must have $c \in (-\sqrt{4/3}, \sqrt{4/3})$. Hence if $f : [a,b] \to [a,b]$ is a contraction, then $[a,b] \subset (-\sqrt{4/3}, \sqrt{4/3})$.