Math 140B. Foundations of Analysis

Lecture 11: Bounded variation.

Recall: A **partition** of an interval $[a, b]$ is a list of points $x_0, \ldots, x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$. We write $\Delta x_k = x_k - x_{k-1}$. if $f : [a, b] \to \mathbb{R}$ we write $\Delta f_k = f(x_k) - f(x_{k-1})$ and

$$\sum(P) \;=\; \sum_a^b(P)(f) \;=\; \sum_{k=1}^n |\Delta f_k|.$$

We define

$$V_f(a, b) \;=\; \sup\left\{\sum(P) : P \text{ is a partition of } [a, b]\right\}.$$

Theorem. *Let f be of bounded variation on $[a, b]$. Let $V(x) = V_f(a, x)$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.*

Proof. For $x \in (a, b)$ we know that $V(x+)$, $V(x-)$, $f(x+)$ and $f(x-)$ exist.
Difficult case: f continuous at c implies V is continuous at c. Assume that $c \in [a, b)$ is a point of continuity of $f(x)$. We will show that $V(c+) = V(c)$. The proof that $V(c-) = V(c)$ for $c \in (a, b]$ is similar.
Given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Now choose a partition $P = (x_0, \ldots, x_n)$ on $[c, b]$ such that

$$V_f(c, b) \;\le\; \sum_c^b(P)(f) + \varepsilon.$$

Now fix x with $c < x < \min\{c + \delta, x_1\}$. Set $P' = (x_0, x, x_1, \ldots, x_n)$, and $P'' = (x, x_1, \ldots, x_n)$. Then

$$V_f(c, b) \;\le\; \sum_c^b(P')(f) + \varepsilon.$$

But then

$$\begin{aligned} V(x) - V(c) \;&=\; V_f(c, b) - V_f(x, b) \;\le\; V_f(c, b) \;-\; \sum_x^b(P'')(f) \\ &\le\; \sum_c^b(P')(f) \;-\; \sum_x^b(P'')(f) \;+\; \varepsilon \;=\; |f(x) - f(c)| + \varepsilon \;\le\; 2\varepsilon. \end{aligned}$$

Hence we see that $V(c+) = V(c)$. Similarly $V(c-) = V(c)$.

Corollary. *Let f be continuous on $[a, b]$. Then f is of bounded variation on $[a, b]$ if and only if it can be expressed as the difference of two increasing continuous functions.*

Proof. Suppose f is continuous and set $V(x) := V_f(a, x)$, then $f = V - (V - f)$, and V and $V - f$ are increasing. But f continuous implies $V - f$ continuous. The converse is clear.

Quiz. Which of the following conditions implies that $f : [a, b] \to \mathbb{R}$ has bounded variation?
(a). f is decreasing.
(b). f is continuous.
(c). f is a product of functions of bounded variation.
(d). $|f|$ is of bounded variation.

Problem 4.68 (a). Suppose that (S, d) is a complete metric space and $f : S \to S$ is a contraction, that is there exists α with $0 \leq \alpha < 1$ such that

$$d(f(x), f(y)) \;\leq\; \alpha d(x, y), \qquad\qquad \text{for all } x, y \in S.$$

Set $p_0 = x$, $p_n = f(p_{n-1}) = f^n(x)$. Show that there exists $p \in S$ such that

$$d(p, p_n) \;\leq\; \frac{\alpha^n}{1-\alpha} d(x, f(x)).$$

Solution.

$$d(p_n, p_{n-1}) \;=\; d(f(p_{n-1}), f(p_n)) \;\leq\; \alpha d(p_{n-1}, p_n).$$

By induction this gives

$$d(p_n, p_{n-1}) \;\leq\; \alpha^n d(p_0, p_1) \;=\; \alpha^n d(x, f(x)).$$

Then for $k \geq 1$,

$$\begin{aligned} d(p_n, p_{n+k}) \;\leq\; d(p_n, p_{n+1}) + \ldots + d(p_{n+k-1}, p_{n+k}) \;&\leq\; (\alpha^n + \cdots + \alpha^{n+k-1}) d(x, f(x)) \\ \leq\; \alpha^n(1 + \alpha + \alpha^2 + \ldots) d(x, f(x)) \;&=\; \frac{\alpha^n}{1-\alpha} d(x, f(x)). \end{aligned}$$

Since the right hand side converges to zero as $n \to \infty$, we see that p_n is Cauchy and hence converges to some $p \in S$ since S is complete. Also, taking the limit as $k \to \infty$ gives

$$d(p_n, p) \;\leq\; \frac{\alpha^n}{1-\alpha} d(x, f(x)).$$

In fact, p is the unique fixed point of f.

Problem 4.68 (b). Take $f(x) = \frac{1}{2}(x + 2/x)$, $S = [1, \infty)$. Prove that f is a contraction of S with contraction constant $\alpha = \frac{1}{2}$ and fixed point $p = \sqrt{2}$. Form the sequence p_n with $p_0 = x = 1$ and show that $|p_n - \sqrt{2}| \leq 2^{-n}$.

Solution. First we must show that $f : [1,\infty) \to [1,\infty)$. Notice that $\lim_{x\to\infty} f(x) = +\infty$. Hence there exists N such that $f(x) > f(1)$ for $x \geq N$. By compactness the minimum value of f on $[1,\infty)$ is attained on $[1,N] \subset [1,\infty)$. Now f is differentiable on $[1,\infty)$ and so if the minimum of f is attained at $c \in (1,\infty)$ then $f'(c) = 0$. We have

$$f'(x) \;=\; \frac{1}{2} - \frac{1}{x^2},$$

which vanishes at $x = \sqrt{2}$. Hence the minimum value of f occurs either at $x = 1$ or at $x = \sqrt{2}$. But $f(1) = 3/2 > 1$ and $f(\sqrt{2}) = \sqrt{2} > 1$, so the minimum of f on $[1,\infty)$ is greater than 1 and $f[1,\infty) \subset [1,\infty)$.

Next we see that f is a contraction on $[1,\infty)$. For this we notice that on $[1,\infty)$ we have

$$|f'(x)| \;=\; \left|\frac{1}{2} - \frac{1}{x^2}\right| \;\leq\; \frac{1}{2}.$$

Hence by the mean value theorem, given $x, y \in [1,\infty)$ there exists θ between x and y such that

$$|f(x) - f(y)| \;=\; |f'(\theta)||x-y| \;\leq\; \frac{1}{2}|x-y|.$$

Now from part (a), we get

$$|p_n - \sqrt{2}| \;\leq\; \frac{(1/2)^n}{1-1/2}|1 - f(1)| \;=\; 2^{-n}.$$

Remark. The function f in this example comes from the Newton-Raphson method of solving $x^2 - 2 = 0$. In general to solve $h(x) = 0$, if x_n is an estimate for the root then

$$x_{n+1} \;=\; x_n \;-\; \frac{h(x_n)}{h'(x_n)}$$

is the next estimate. In the case $h(x) = x^2 - 2$ we have

$$x \;-\; \frac{h(x)}{h'(x)} \;=\; x \;-\; \frac{x^2-2}{2x} \;=\; \frac{1}{2}\left(x + \frac{2}{x}\right).$$