MATH 140B. FOUNDATIONS OF ANALYSIS

Lecture 12: Bounded variation.

Recall: A **partition** of an interval $[a, b]$ is a list of points $x_0, \dots, x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$. We write $\Delta x_k = x_k - x_{k-1}$. if $f : [a, b] \to \mathbb{R}$ we write $\Delta f_k = f(x_k) - f(x_{k-1})$ and

$$\sum(P) \;=\; \sum_a^b(P)(f) \;=\; \sum_{k=1}^n |\Delta f_k|.$$

The partition $P' = (y_0, y_1, \dots, y_m)$ is **finer than** P, written $P \subset P'$, if

$$\{x_0, \dots, x_n\} \;\subset\; \{y_0, \dots, y_m\}.$$

In general for partitions $P = (x_0, \dots, x_n)$, $P' = (y_0, \dots, y_m)$ and $P'' = (z_0, \dots, z_k)$, we write $P = P' \cup P''$ if

$$\{x_0, \dots, z)n\} \;=\; \{y_0, \dots, y_m\} \cup \{z_0, \dots, z_k\}.$$

Remark. In what follows we assume that a, b are finite and all functions $\alpha, f : [a, b] \to \mathbb{R}$ are **bounded**.

Definition. For the partition P, suppose $t_k \in [x_{k-1}, x_k]$.

$$S(P, f, \alpha) \;=\; \sum_{k=1}^n f(t_k)\Delta\alpha_k$$

is called a **Riemann-Stieltjes sum of** f **with respect to** α.

We say f is **Riemann integrable with respect to** α **on** $[a, b]$ and we write "$f \in R(\alpha)$ on $[a, b]$" or equivalently "$\int_a^b f\, d\alpha$ exists" if:
There exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that for every partition P of $[a, b]$ which is finer than P_ε and for every choice $t_k \in [x_{k-1}, x_k]$, we have

$$|S(P, f, \alpha) - A| < \varepsilon.$$

In this case we write

$$\int_a^b f\, d\alpha \;=\; A.$$

Remark. We sometimes write $\int_a^b f\, d\alpha$ as $\int_a^b f(x)\, d\alpha(x)$. The symbol x is a dummy variable which serves only to emphasize the fact that f and α are both functions on $[a, b]$.

We will show that the Riemann-Stieltjes integral satisfies several properties if it exists, and then give conditions on f and α such that it does exist.

We prove Theorem 7.2, 7.3 and 7.4.

Theorem. *If $\int_a^b f\, d\alpha$ exists and $c \in (a, b)$ then $\int_a^c f\, d\alpha$ exists.*