Lecture 14: Integration by parts.

Theorem 7.6. *If $f \in R(\alpha)$ on $[a,b]$, then $\alpha \in R(f)$ on $[a,b]$ and*

$$\int_a^b f(x)\,d\alpha(x) \;+\; \int_a^b \alpha(x) df(x) \;=\; f(b)\alpha(b) \;-\; f(a)\alpha(a).$$

Proof. There is a picture associated with this result!
Assume $f \in R(\alpha)$ and given $\varepsilon > 0$, choose a partition $P(\varepsilon)$ such that

$$P \supset P(\varepsilon) \qquad \Rightarrow \qquad \left|\sum_P f(t_j)(\alpha(x_j) - \alpha(x_{j-1}) - \int_a^b f(x)\,d\alpha(x)\right| < \varepsilon.$$

Now we consider such a partition P with points $t_j \in [x_{j-1}, x_j]$, and for convenience set $t_0 = a$ and $t_{n+1} = b$. Then

$$\begin{aligned}
S(P,\alpha,f) \;&=\; \sum_{j=1}^{n} \alpha(t_j)(f(x_j) - f(x_{j-1}) \;=\; -\sum_{j=1}^{n} \alpha(t_j)(f(x_{j-1}) - f(x_j))\\
&=\; -\left(\sum_{j=1}^{n} \alpha(t_j) f(x_{j-1}) - \sum_{j=2}^{n+1} \alpha(t_{j-1}) f(x_{j-1})\right)\\
&=\; -\left(\sum_{j=1}^{n+1} \alpha(t_j) f(x_{j-1}) - \sum_{j=1}^{n+1} \alpha(t_{j-1}) f(x_{j-1})\right) \;+\; \alpha(t_{n+1}) f(x_n) \;-\; \alpha(t_0) f(x_0)\\
&=\; -\sum_{j=1}^{n+1} f(x_{j-1})(\alpha(t_j) - \alpha(t_{j-1})) \;+\; f(b)\alpha(b) \;-\; f(a)\alpha(a)\\
&=\; \sum_{j=1}^{n+1} \big(f(x_{j-1})(\alpha(t_j) - \alpha(x_{j-1}) + f(x_{j-1})(\alpha(x_{j-1}) - \alpha(t_{j-1}))\big) \;+\; f(b)\alpha(b) \;-\; f(a)\alpha(a)\\
&=\; -S(P', f, \alpha) \;+\; f(b)\alpha(b) \;-\; f(a)\alpha(a),
\end{aligned}$$

where $P' = \{x_0, t_1, x_1, t_2, \ldots, t_{n-1}, x_n\}$ and the new "t_j" points are the points $x_{j-1} \in [t_{j-1}, x_{j-1}]$ and $x_{j-1} \in [x_{j-1}, t_j]$. Since P' is a refinement of P and hence of $P(\varepsilon)$, we see that

$$\left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right| < \varepsilon,$$

and hence since

$$S(P', \alpha, f) \;-\; \int_a^b f\,d\alpha \;=\; -S(P, f, \alpha) \;+\; f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha,$$

we get

$$\left|S(P,f,\alpha) \;-\; \left(f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha\right)\right| \;<\; \varepsilon,$$

and we get the conclusion.

Theorem 7.7. *Let g be a strictly increasing continuous function defined on an interval $[c,d]$ and suppose $f \in R(\alpha)$ on the interval $[g(c), g(d)]$. Then $f \circ g \in R(\alpha \circ g)$ on $[c,d]$ and*

$$\int_{g(c)}^{g(d)} f(t)\,d\alpha(t) \;=\; \int_c^d f(g(x))\,d(\alpha(g(x))).$$

Proof. To be completed by the audience.