Math 140B. Foundations of Analysis

Lecture 18: Lebesgue's criterion.

Last time: If α is increasing then the following are equivalent:
(i) $f \in R(\alpha)$ on $[a, b]$.
(ii) f satisfies Riemann's condition with respect to α on $[a,]$.
(iii) $\underline{\int} f \, d\alpha = \overline{\int} f \, d\alpha$.

We need to assume (iii) and prove (i). Set $I = \underline{\int} f \, d\alpha = \overline{\int} f \, d\alpha$. Given $\varepsilon > 0$, there exists a partition P_ε such that

$$P \supset P_\varepsilon \qquad \Rightarrow \qquad L(P, f, \alpha) > I - \varepsilon,$$

and there exists P'_ε such that

$$P \supset P'_\varepsilon \qquad \Rightarrow \qquad U(P, f, \alpha) < I + \varepsilon.$$

But then

$$P \supset P_\varepsilon \cup P'_\varepsilon \qquad \Rightarrow \qquad I - \varepsilon < L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha) < I + \varepsilon$$
$$\Rightarrow \qquad |S(P, f, \alpha) - I| < \varepsilon.$$

Hence $f \in R(\alpha)$ on $[a, b]$.

Definition. We say $f \in R$ on $[a, b]$ if $\int_a^b f(x) \, dx$ exists.

Quiz. For which of the following functions does $\int_0^1 f(x) \, dx$ exist?

(a) $$f(x) = \exp \sin \left(\frac{1}{x+1} \right)$$

(b) $$f(x) = [100x]$$

(c) $$f(x) = \frac{1}{\sqrt{x}}$$

(d) $$f(x) = x \sin(1/x).$$

Recall. If f is continuous on $[a, b]$ and α is of bounded variation then $f \in R(\alpha)$.

Today we ask: give examples when f is not integrable with respect to α on $[a, b]$.

Theorem 7.29. *Assume α is increasing on $[a, b]$ and $a < c < b$. Assume further that both α and f are both discontinuous from the right at $x = c$ (or both discontinuous from the left). Then $\int_a^b f \, d\alpha$ does not exist.*

Proof. Let P be a partition of $[a, b]$ containing c, so $x_{j-1} = c$ for some j.
(*)

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{k=1}^{n} (M_k(f) - m_k(f)) \Delta_k \alpha \geq (M_j(f) - m_j(f))(\alpha(x_j) - \alpha(c))$$

But since f is discontinuous from the right at c, there exists $\varepsilon > 0$ such that $M_j(f) - m_j(f) > \varepsilon$ for any choice of the next partition point x_j, and there exists $\delta > 0$ such that $\alpha(x_j) - \alpha(c) > \delta$ regardless of x_j. Hence (*) is bounded below by $\varepsilon\delta$ and so Riemann's condition is not satisfied.

we cover
Definition 7.43.
Theorem 7.44.
The statement of Theorem 7.48.
Definition 7.45.
Theorem 7.47