Math 140B. Foundations of Analysis

Lecture 3. Chain rule, one sided limits and derivatives, vanishing derivatives.

Last time: $f : (a,b) \to \mathbb{R}$, then f is differentiable at $c \in (a,b)$ if

$$f'(c) \ := \ \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In this case, the function

$$f^*(x) \ = \ \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

is continuous on (a,b) and

$$f(x) - f(c) \ = \ f^*(x)(x-c), \qquad \text{for all } x \in (a,b).$$

Theorem. *(Chain Rule) Let f be a function defined on an open interval S, let g be defined on an open interval containing $f(S)$, and consider the composite function $g \circ f$ on S given by*

$$(g \circ f)(x) \ = \ g(f(x)).$$

If f is differentiable at $c \in S$ and g is differentiable at $g(c)$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) \ = \ g'(f(c)) \ f'(c).$$

Proof. We define f^* as above and set

$$g^*(y) \ = \ \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c). \end{cases}$$

Then g^* is continuous at $f(c)$, and

$$g(y) - g(f(c)) \ = \ g^*(y)(y - f(c)).$$

Hence

$$(g \circ f)(x) - (g \circ f)(c) \ = \ g^*(f(x))(f(x) - f(c)) \ = \ (g^* \circ f)(x) \ f^*(x) \ (x-c).$$

For $x \neq c$, we get

$$(*) \qquad \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} \ = \ (g^* \circ f)(x) \ f^*(x).$$

Quiz. Explain why the right hand side tends to $g'(f(c))f'(c)$ as $x \to c$.

One sided limits. For $f : (a, b) \to \mathbb{R}$ and $c \in [a, b)$, we write

$$\lim_{x \to c^+} f(x) = A, \qquad \text{or equivalently} \qquad f(c^+) = A,$$

if

$$\lim_{x \to c} f|_{(c,b)}(x) = A,$$

where $f|_{(c,b)}$ is the function f restricted to the interval (c, b). This is saying that $f(x) \to A$ as $x \to c$ through values greater than c. Equivalently:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$c < x < c + \delta \qquad \Rightarrow \qquad |f(x) - A| < \varepsilon.$$

Quiz. Define

$$\lim_{x \to c-} f(x) = A, \qquad \text{or equivalently} \qquad f(c-) = A.$$

Remark. $f(c-)$ and $f(c+)$ need not be equal. $f(c+) - f(c-)$ is called the **jump of f at c.**

Quiz. (a). Describe when $\lim_{x \to c} f(x)$ exists in terms of $f(c-)$ and $f(c+)$.

(b). Describe when f is continuous at c in terms of $f(c-)$ and $f(c+)$.

Remark. If $f(c-) = f(c+) \neq f(c)$, then we say f has a **removable singularity** at c. We can redefine $f(c)$ to obtain a continuous function.

Theorem. *If $f : (a, b) \to \mathbb{R}$ is increasing and $c \in (a, b)$, then $f(c-)$ and $f(c+)$ both exist and*

$$f(c-) \leq f(c) \leq f(c+).$$

Proof. Let $A = \{f(x) : a < x < c\}$. Since f is increasing, this set is bounded above by $f(c)$ and hence has a supremum. Let $\alpha = \sup A$. Then $\alpha \leq f(c)$. We will show that $f(c-)$ exists and equals α. Because α is the supremum of A, given $\varepsilon > 0$, there exists $\delta > 0$ with $a < c - \delta < c$ such that

$$\alpha - \varepsilon \; < \; f(c - \delta).$$

But then

$$c - \delta < x < c \qquad \Rightarrow \qquad \alpha - \varepsilon \; < \; f(c - \delta) \; \leq \; f(x) \; \leq \; \alpha \qquad \Rightarrow \qquad |f(x) - \alpha| < \varepsilon.$$

Hence $f(c-)$ exists and equals α. The proof for $f(c+)$ is similar.

Definition. Let f be defined on a closed interval S and assume that f is continuous at $c \in S$. Then f is said to have a **righthand derivative** at c if

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

exists, or if the limit is $+\infty$ or $-\infty$. The limit is denoted by $f'_+(c)$. The **lefthand derivative** $f'_-(c)$ is defined similarly.

Quiz. When is f differentiable at c in terms of $f'_-(c)$ and $f'_+(c)$?

Theorem. *Suppose* $f : (a, b) \to \mathbb{R}$ *is differentiable at* $c \in (a, b)$ *and* $f'(c) > 0$ *(or* $f'(c) = +\infty$*). Then there exists* $\delta > 0$ *such that*

$$f(x) > f(c) \quad \text{if } c < x < c + \delta, \qquad\qquad f(x) < f(c) \qquad \text{if } c - \delta < x < c.$$

Proof. Since f is differentiable at c, The function

$$f^*(x) \;=\; \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

is continuous at c and we are assuming $f^*(c) = f'(c) > 0$. But then by continuity, there exists $\delta > 0$ such that $f^*(c) > 0$ on $(c - \delta, c + \delta)$. From the definition of f^*, we get the result.