Math 140B. Foundations of Analysis

Lecture 4: Limits of sequences.

Error last time: Correct definition:

$$f'_+(c) \;=\; \lim_{x\to c+} \frac{f(x)-f(c)}{x-c}, \qquad\qquad f'(c+) = \lim_{x\to c} f'(x).$$

If the right hand side exists then so does the left hand side and they are equal (problem 5.16). However, the left hand side can exist without the right existing, for example

$$f(x) \;=\; x^2 \sin\frac{1}{x}.$$

Then we see that $f'(x)$ exists for all $x \in \mathbb{R}$, but $f'(x)$ is discontinuous at $x = 0$. Indeed, $f'(0) = 0$, and

$$f'(x) \;=\; 2x\sin\frac{1}{x} \;-\; \sin\frac{1}{x},$$

and this diverges as $x \to 0$.

Theorem. *(Dominated convergence theorem for series.) Suppose that for $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$, we have a sequence of complex number $a_{k,n}$, and suppose that for each fixed k,*

(1) $$\lim_{n\to\infty} a_{k,n} \qquad \textit{exists},$$

and suppose there exists a sequence $b_k \geq 0$ such that

(2) $$|a_{k,n}| \;\leq\; b_k, \qquad\qquad \textit{for all } k \textit{ and } n.$$

and

(3) $$\sum_k b_k \;<\; \infty$$

Then

$$\sum_k \lim_{n\to\infty} a_{k,n} \;=\; \lim_{n\to\infty} \sum_k a_{k,n},$$

and the left hand side converges absolutely.

Proof. Set $a_k = \lim_{n\to\infty} a_{k,n}$. Then (1) and (2) imply that $|a_k| \leq b_k$, and so $\sum a_k$ converges by the squeeze test. Given $\varepsilon > 0$, there exists N such that

$$\sum_{N+1}^{\infty} b_k \;<\; \frac{\varepsilon}{3}.$$

But then then there exists $N' \geq N$ such that

$$n \geq N' \qquad \Rightarrow \qquad \sum_{k \leq N} |a_{k,n} - a_k| \; < \; \frac{\varepsilon}{3}.$$

Then $n \geq N' \;\Rightarrow$

$$\left| \sum_k a_{k,n} - \sum_k a_k \right| \;\leq\; \left| \sum_{k \leq N} a_{k,n} - \sum_{k \leq N} a_k \right| \;+\; \left| \sum_{k=N+1}^{\infty} a_{k,n} \right| \;+\; \left| \sum_{k=N+1}^{\infty} a_k \right|$$

$$\leq \; \sum_{k \leq N} |a_{k,n} - a_k| \;+\; 2 \sum_{k=N+1}^{\infty} b_k \;<\; \varepsilon.$$

Corollary.

$$\lim_{n \to \infty} (1 + 1/n)^n \;=\; e.$$

Proof.

$$(1+1/n)^n \;=\; 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \frac{1}{n^2} + \ldots + \frac{n(n-1)\ldots(n-k+1)}{k!} \frac{1}{n^k} + \ldots + \frac{1}{n^n}$$

$$= \; 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \ldots \left(\frac{1}{n}\right)$$

$$= \; \sum_{k=0}^{\infty} a_{k,n},$$

where

$$a_{k,n} \;=\; \begin{cases} \frac{1}{k!}\left(1 - \frac{1}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) & k \leq n \\ 0 & k > n. \end{cases}$$

Now clearly for each fixed n we have $\lim_{n \to \infty} a_{k,n} = \frac{1}{k!}$. But

$$|a_{k,n}| \;\leq\; \frac{1}{k!}, \qquad \sum_k \frac{1}{k!} < \infty.$$

Hence by dominated convergence,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \;=\; \lim_{n \to \infty} \sum_{k=0}^{n} a_{k,n} \;=\; \sum_{k=0}^{\infty} \lim_{n \to \infty} a_{k,n} \;=\; \sum_{k=0\infty} \frac{1}{k!} \;=\; e.$$

Theorem 1'. *(Dominated convergence for series.) Suppose we have a sequence of functions $f_k : (a,b) \to \mathbb{R}$, and suppose that for each k,*

$$\lim_{x \to a} f_k(x) \qquad \textit{exists.} \tag{1}$$

Suppose moreover that there exists a non-negative real sequence b_k such that

$$|f_k(x)| \;\le\; b_k \qquad \textit{for all } k = 1, 2, \ldots \textit{ and } x \in (a,b) \tag{2}$$

and that

$$\sum b_k \;<\; \infty.$$

Then

$$\lim_{x \to a} \sum_k f_k(x) \;=\; \sum_k \lim_{x \to a} f_k(x).$$

Quiz Write a proof.

Corollary. *If $f(x) = \sum_0^\infty a_n x^n$ is a power series with radius of convergence R, then $g(x) = \sum_1^\infty a_n n x^{n-1}$ also has radius of convergence R and $f(x)$ is differentiable on $(-R, R)$ with $f'(x) = g(x)$.*

Quiz. Why does the power series g have the same radius of convergence as the power series f?

Proof.

Suppose $x \in (-R, R)$ and choose ρ with $0 < |x| < \rho < R$. Then if $|y| < \rho$ we have

$$\left| \frac{y^n - x^n}{y - x} \right| \;=\; \left| y^{n-1} + x y^{n-2} + \cdots + x^{n-1} \right| \;\le\; n \rho^{n-1}.$$

But since $\rho < R$, the sum $\sum_n a_n n \rho^{n-1}$ converges absolutely. Hence by the dominated convergence theorem,

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \;=\; \lim_{y \to x} \sum_n a_n \frac{y^n - x^n}{y - x} \;=\; \sum_n a_n \lim_{n \to \infty} \frac{y^n - x^n}{y - x} \;=\; \sum_n a_n n x^{n-1}.$$

Corollary. $\exp(x)$ *is differentiable with derivative* $\exp(x)$*.*

Theorem. *If $f : (a,b) \to (a,d)$ is a homeomorphism and f is differentiable at c, then f^{-1} is differentiable at $f(c)$ and*

$$(f^{-1})'(c) \;=\; \frac{1}{f'(c)}.$$

Corollary. *$f(x) = \log x$ then $f'(x) = 1/x$.*

Another proof that $(1+1/n)^n \to e$. Set

$$f^*(x) \;=\; \begin{cases} \frac{\log(1+x)-0}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

Then $f^*(x)$ is continuous at $x = 0$, and

$$\log(1+1/n) \;=\; \frac{f^*(1/n)}{n}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty}(1+1/n)^n \;&=\; \lim_{n\to\infty} \exp\left(n \log(1+1/n)\right) \\ &=\; \lim_{n\to\infty} \exp(f^*(1/n)) \;=\; \exp\left(\lim_{n\to\infty} f^*(1/n)\right) \;=\; \exp(1). \end{aligned}$$