Lecture 6: Taylor's formula.

Definition. We say that a function $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ if f is differentiable on (a, b), right differentiable at a and left differentiable at b (that is f' exists on (a, b) and $f'_+(a)$ and $f'_-(b)$ exist). If the function f' is differentiable, then we denote the derivative of f' by f'' and call it the second derivative of f. We can continue this process to define $f^{(n)}$, the nth derivative of f on $[a, b]$.

Theorem 5.19. *(Taylor's formula) Suppose $f : [a, b] \to \mathbb{R}$ satisfies*

- *f is n-times differentiable on (a, b),*
- *f is $(n-1)$-times differentiable on $[a, b]$,*
- *$f^{(n-1)}$ is continuous on $[a, b]$.*

Then if c and x are distinct points in $[a, b]$ with $x \neq c$, there exists a point x_1 lying strictly between x and c such that

$$f(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k \;+\; \frac{f^{(n)}(x_1)}{n!}(x-c)^n.$$

Remarks. 1. In this formula, we make the convention that $0^0 = 1$, so that the first term in the sum on the right at $x = c$ is $f(c)$.

2. The conditions of the theorem are certainly satisfied if f is n-times differentiable on $[a, b]$.

3. Notice that if we define

$$P(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

then one can check by differentiating that

$$P(c) = f(c), \qquad P'(c) = f'(c), \qquad \ldots \qquad , P^{(n-1)}(c) = f^{(n-1)}(c).$$

The polynomial $P(x)$ is the unique $(n-1)-st$ order polynomial whose value and first $n-1$ derivatives agree with those of f.