MATH 140B. FOUNDATIONS OF ANALYSIS

Lecture 7: Taylor's formula.

Last time:

Theorem. *(Generalized Mean Value Theorem.) Suppose* $f : [a,b] \to \mathbb{R}$ *and* $g : [a,b] \to \mathbb{R}$ *are both differentiable on* (a,b) *and continuous on* $[a,b]$*. Then there exists a point* $c \in (a,b)$ *such that*

$$f'(c)(g(b) - g(a)) \;=\; g'(c)(f(b) - f(a)).$$

Theorem 5.19. *(Taylor's formula) Suppose* $f : [a,b] \to \mathbb{R}$ *satisfies*

- f *is* n*-times differentiable on* (a,b)*,*
- f *is* $(n-1)$*-times differentiable on* $[a,b]$*,*
- $f^{(n-1)}$ *is continuous on* $[a,b]$*.*

Then if c *and* x *are distinct points in* $[a,b]$ *with* $x \neq c$*, there exists a point* x_1 *lying strictly between* x *and* c *such that*

$$f(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k \;+\; \frac{f^{(n)}(x_1)}{n!}(x-c)^n.$$

Remarks. 1. In this formula, we make the convention that $0^0 = 1$, so that the first term in the sum on the right at $x = c$ is $f(c)$.

2. The conditions of the theorem are certainly satisfied if f is n-times differentiable on $[a,b]$.

3. Notice that if we define

$$P(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

then one can check by differentiating that

$$P(c) = f(c), \qquad P'(c) = f'(c), \qquad \ldots \qquad , P^{(n-1)}(c) = f^{(n-1)}(c).$$

The polynomial $P(x)$ is the unique $(n-1)-st$ order polynomial whose value and first $n-1$ derivatives agree with those of f.

Proof of Taylor's formula. First note the strategy:

- We are going to apply the generalized mean value theorem. For now, let us take any other function $g : [a,b] \to \mathbb{R}$ which satisfies the same conditions as f. Later we will take $g(x) = (x-c)^n$.
- We are going to apply the generalized mean value theorem not to a function of x, but to a function of c.
- For simplicity we assume $c < x$.

For $t \in [c, x]$, we define

$$F(t) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k, \qquad\qquad G(t) \;=\; \sum_{k=0}^{n-1} \frac{g^{(k)}(t)}{k!}(x-t)^k.$$

Then F and G are continuous on $[c, x]$ and differentiable on (c, x). Hence by the generalized mean value theorem, there exists $x_1 \in (c, x)$ such that

$$(*) \qquad F'(x_1)(G(x) - G(c)) \;=\; G'(x_1)(F(x) - F(c)).$$

Now we evaluated this in terms of f and g. First note that $F(x) = f(x)$ and $G(x) = g(x)$. Furthermore, using the product rule,

$$\begin{aligned} F'(t) \;&=\; \sum_{k=0}^{n-1} \left(\frac{f^{(k+1)}(t)}{k!}(x-t)^k \;-\; \frac{f^k(t)}{(k-1)!}(x-t)^{k-1} \right) \\ &=\; \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \;-\; \sum_{k=1}^{n-1} \frac{f^k(t)}{(k-1)!}(x-t)^{k-1} \\ &=\; \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}. \end{aligned}$$

Hence $(*)$ becomes

$$\begin{aligned} &\frac{f^{(n)}(x_1)}{(n-1)!}(x-x_1)^{n-1} \left(g(x) \;-\; \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(c-t)^k \right) \\ &\qquad =\; \frac{g^{(n)}(x_1)}{(n-1)!}(x-x_1)^{n-1} \left(f(x) \;-\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(c-t)^k \right). \end{aligned}$$

Clearing common factors, we get

$$\begin{aligned} (**) \quad &f^{(n)}(x_1) \left(g(x) \;-\; \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(c-t)^k \right) \\ &\qquad =\; g^{(n)}(x_1) \left(f(x) \;-\; \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(c-t)^k \right). \end{aligned}$$

This formula is more general than Taylor's formula. Setting $g(x) = (x-c)^n$ we have

$$g(x) \;-\; \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(c-t)^k \;=\; (x-c)^n,$$

and

$$g^{(n)}(x) \;=\; n!.$$

Hence Taylor's formula follows.

Example. Use Taylor polynomials to calculate $\sqrt{2}$ to an accuracy of $1/20$. Can they be used to get an accuracy of $1/10^6$?

Solution. We approximate $\sqrt{2}$ by the $(n-1)$st order Taylor's polynomial of $f(x) = \sqrt{x}$ around $c = 1$.

$$\frac{1}{n!} \sup_{x_1 \in [1,2]} |f^{(n)}(x_1)|.$$

Now on $[1,2]$ we have

$$|f(x)| = |\sqrt{x}| \leq 2, \qquad |f'(x)| = \left|\frac{x^{-1/2}}{2}\right| \leq \frac{1}{2}, \qquad |f''(x)| = \left|\frac{x^{-3/2}}{4}\right| \leq \frac{1}{4},$$

$$|f^{(3)}(x)| = \left|\frac{3x^{-5/2}}{8}\right| \leq \frac{3}{8}, \qquad |f^{(k)}(x)| \leq \frac{1 \cdot 3 \cdots (2k-3)}{k! 2^k} := \varepsilon_k.$$

Hence

$$\varepsilon_1 = \frac{1}{2}, \qquad \varepsilon_2 = \frac{1}{8}, \qquad \varepsilon_3 = \frac{3}{2 \cdot 3 \cdot 2^3} = \frac{1}{16}, \qquad \varepsilon_4 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^4} = \frac{5}{128}.$$

Hence

$$\begin{aligned} \sqrt{2} \;&=\; 1 + f'(0) + \frac{1}{2!} f''(0) 1^2 + \frac{1}{3!} f^{(3)}(0) 1^3 + \left(\text{error bounded by } \frac{5}{128} < \frac{1}{20}\right) \\ &=\; 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} + \left(\text{error bounded by } \frac{5}{128} < \frac{1}{20}\right). \end{aligned}$$

Now write

$$p_n(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!} (x-1)^k.$$

Then we ask whether

$$|\sqrt{2} - p_n(2)| \;\to\; 0, \qquad \text{as } n \to \infty.$$

For those who know complex analysis, we remark that the function $z \to \sqrt{z}$ can be extended to an analytic function on $\mathbb{C} \setminus \{z \leq 0\}$, and it has a singularity at $z = 0$. Hence the radius of convergence of the power series for $\sqrt{z}$ at $z = 1$ equals 1, and the radius of convergence of $\sqrt{z}$ at $z = 4$ equals 4. In the former case 2 lies on the boundary of the convergent region while in the later case it lies within the radius of convergence. We might expect a better rate of convergence by taking $c = 4$. However, continuing with $c = 1$, we already have the bound

$$\begin{aligned} |\sqrt{2} - p_n(2)| \;\leq\; \frac{\varepsilon_n}{n!} \;&=\; \frac{1 \cdot 3 \cdots (2n-3)}{n! 2^n} \;=\; \frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n)} \\ &=\; \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-3}{2n-2} \cdot \frac{1}{2n} \;<\; \frac{1}{2n} \;\to\; 0 \text{ as } n \to \infty. \end{aligned}$$

Technical example of what Taylor's formula can do for you. Show that

$$(*) \qquad (1+1/n)^n \;=\; \left(1-\frac{1}{2n}+\frac{\psi(n)}{n^2}\right)e,$$

where

$$0<\psi(n)<\frac{1}{3}+\frac{1}{8}.$$

This gives a very good estimate for the mysterious function $(1+1/n)^n$ for large values of n. We can use Taylor's formula to get even better estimates if we have the stamina!