MATH 140B. FOUNDATIONS OF ANALYSIS

Lecture 8: Differentiation in $\mathbb{R}^n$.

Remark from last time: Approximating $\sqrt{x}$ by the degree $(n-1)$ Taylor polynomial $p_n(x)$ based at $c = 1$ we have

$$|\sqrt{2} - p_n(2)| \to 0 \qquad \text{as } n \to \infty.$$

This can be seen in two ways. Set

$$c_k = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-3}{2k-2} \cdot \frac{1}{2k}.$$

Then $c_k \to 0$ as $k \to \infty$ since

$$0 < c_k < \frac{1}{2k}.$$

Either notice that $p_n(2)$ is the $(n-1)$st partial sum of the alternating series

$$1 + \frac{1}{2} + \sum_{k=2}^{n-1} (-1)^{k+1} c_k,$$

or use Taylor's formula to see that

$$|\sqrt{2} - p_n(2)| < c_n.$$

Quiz. Let (S, d) be a metric space. Are the following true or False?
(a) If $f : S \to \mathbb{R}^2$ and $p \in S$. Write f in components: $f = (f_1, f_2)$. Then

$$\lim_{x \to p} f(x) = (L_1, L_2) \quad \Leftrightarrow \quad \lim_{x \to p} f_1(x) = L_1 \quad \text{and} \quad \lim_{x \to p} f_2(x) = L_2.$$

(b). Suppose U is an open subset of $\mathbb{R}^n$ and $p = (p_1, p_2) \in U$. Suppose $f : U \to S$. Then

$$\lim_{(x_1, x_2) \to (p_1, p_2)} f(x) = L \quad \Leftrightarrow \quad \lim_{x_1 \to p_1} f(x_1, p_2) = L \quad \text{and} \quad \lim_{x_2 \to p_2} f(p_1, x_2) = L.$$

Definition. Let $f : (a, b) \to \mathbb{R}^n$ be a vector valued function. Write f in components, so $f = (f_1, \ldots, f_n)$. Then we say f is differentiable at the point $c \in (a, b)$ if

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists, equivalently if

$$f_j'(c) = \lim_{x \to c} \frac{f_j(x) - f_j(c)}{x - c}$$

exists for $j = 1, 2, \ldots, n$.

The usual rules for derivatives extend as you would expect, for example the chain rule for compositions $(a, b) \to (c, d) \to \mathbb{R}^n$, and the product rule for the scalar product: $(f \cdot g)' = f' \cdot g + f \cdot g'$ etc..

Definition. If Ω is an open set in $\mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \in \Omega$, then the kth partial derivative of f at c is defined to be

$$D_k f(c) \ := \ \lim_{x_k \to c_k} \frac{f(c_1, \ldots, c_{k-1}, x_j, c_{k+1}, \ldots, c_n) \ - \ f(c)}{x_k - c_k},$$

when this limit exists.
Warning: The existence of all partial derivatives at a point does not even imply continuity at the point, for example consider $c = (0, 0)$ and the function

$$f(x, y) \ = \ \begin{cases} 1 & xy \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Even the existence of partial derivatives everywhere does not ensure continuity, for example

$$f(x, y) \ = \ \frac{xy}{x^2 + y^2}$$

has partial derivatives everywhere but is not continuous at $(0, 0)$.

Definition. A function $f : \Omega \to \mathbb{R}$ is differentiable at c if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{x \to c} \frac{f(x) - f(c) - A(x - c)}{|x - c|} \ = \ 0$$

If such an A exists then the partial derivatives of f at c exist, and $D_k f(c) = Ae_k$. We write $Df(c) = Df|_c := A$. It is easy to show that differentiability at c implies continuity at c.

Remark. The linear map A is multiplication by a matrix.

$$A(x - c) \ = \ (A_1 \ldots A_n) \begin{pmatrix} x_1 - c_1 \\ \vdots \\ x_n - c_n \end{pmatrix} \ = \ A_1(x_1 - c_1) + \cdots + A_n(x_n - c_n).$$

Example. If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0, 0)$, show that the partial derivatives exist at $(0, 0)$ and identify the matrix of $Df(0, 0)$ in terms of the partial derivatives of the components of f at $(0, 0)$.

Solution. Write

$$Df|_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \ = \ (\, A_1 \quad A_2 \,),$$

so

$$\lim_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix} - (A_1x_1 + A_2x_2)}{\left|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right|} = 0.$$

Setting $x_2 = 0$ gives

$$\lim_{x_1 \to 0} \frac{f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix} - A_1x_1}{x_1} = 0.$$

Hence $D_1f|_{(0,0)}$ exists and equals A_1. Similarly $D_2f|_{(0,0)} = A_2$.

Definition. If Ω is open in $\mathbb{R}^n$ then a function $f : \Omega \to \mathbb{R}^m$ is differentiable if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to c} \frac{f(x) - f(c) - A(x - c)}{|x - c|} = 0.$$

We write $Df|_c := A$.

Example. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $(0, 0)$, identify the matrix of $Df_{(0,0)}$ in terms of the partial derivatives of the components of f at $(0, 0)$.

Solution. First note that the linear map $A = Df|_{(0,0)}$ is given by multiplication by a 2×2 matrix. We have

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}$$

Write

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \end{pmatrix}.$$

Then

$$\lim_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{\begin{pmatrix} u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} u\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ v\begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}}{\left|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right|} = 0.$$

Taking the first component we get

$$\lim_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - u\begin{pmatrix} 0 \\ 0 \end{pmatrix} - (A_{11}x_1 + A_{12}x_2)}{\left|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right|} = 0.$$

Hence $D_1 u = A_{11}$ and $D_2 u = A_{12}$. Similarly writing the formula for v we get

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{pmatrix}.$$

We remember this by the formula

$$\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \begin{pmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}.$$

A useful theorem in is:

Theorem. *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ is such that the partial derivatives $D_k f(x)$ exist for all $x \in \Omega$ and $k = 1, \ldots, n$. Suppose in addition that each function $D_k f$ is continuous on Ω. Then f is differentiable at each point of Ω and the derivative (matrix) $Df|_x$ is a continuous function of $x \in \Omega$.*