Lecture 12.

Recap: Why do we care about the null space and the column space?

$$A\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{x} \in \text{Nul } A.$$

$$A\mathbf{x} = \mathbf{b} \text{ is consistent} \quad \Leftrightarrow \quad \mathbf{b} \in \text{Col } A.$$

4.5 Dimension Recall that $\mathbf{b}_1, ..., \mathbf{b}_n$ form a **basis** for a vector space V if
(i) $\mathbf{b}_1, ..., \mathbf{b}_n$ are linearly independent, and (ii) $\mathbf{b}_1, ..., \mathbf{b}_n$ span V.

Th If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a spanning set for V then any collection of p vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$, where $p > n$, is linearly dependent.

Pf Since $\mathbf{v}_1, ..., \mathbf{v}_n$ span V we can write each $\mathbf{u}_i$ as a linear combination:

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + ... + a_{in}\mathbf{v}_n$$

A linear combination of the $\mathbf{u}_i$ can be written

$$\begin{aligned} c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p &= c_1\big(a_{11}\mathbf{v}_1 + \cdots + a_{1n}\mathbf{v}_n\big) + \cdots + c_p\big(a_{p1}\mathbf{v}_1 + \cdots + a_{pn}\mathbf{v}_n\big) \\ &= \big(c_1a_{11} + \cdots + c_pa_{p1}\big)\mathbf{v}_1 + \cdots + \big(c_1a_{p1} + \cdots + c_pa_{pn}\big)\mathbf{v}_n \end{aligned}$$

Hence $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$ if

$$\begin{aligned} c_1a_{11} + \cdots + c_pa_{p1} &= 0 \\ &\vdots \\ c_1a_{1n} + \cdots + c_pa_{pn} &= 0 \end{aligned}$$

Since $p > n$ this is a homogenous system for $c_1, \cdots, c_p$ with more unknowns than equations so it has a nontrivial solution. Hence there are constants $c_1, ..., c_p$ not all zero such that $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$, i.e. $\mathbf{u}_1, ..., \mathbf{u}_p$ are linearly dependent.

Cor If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ are bases for V then $n = m$.

The number of elements in a basis for V is called the **dimension** of V, written $\dim V$.

Ex Subspaces of $\mathbf{R}^3$:
1-dimension Span$\{\mathbf{v}\}$ is a line through the origin.
2-dimensions Span$\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}, \mathbf{v}$ are not parallel is a plane through the origin.
3-dimensions Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ are linearly independent is all of $\mathbf{R}^3$, because the columns of $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ span all of $\mathbf{R}^3$ by the Invertible Matrix Theorem.

The Basis Theorem Let V be an n dimensional vector space. Any set of n vectors that spans V is a basis. Any linearly independent set of n vectors in V is a basis.

Dimensions of the column and null spaces of a matrix.

Ex Find $\dim \text{Col } A$ and $\dim \text{Nul } A$, where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$.

Sol $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ so the pivot columns $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ is a basis for Col A and $\dim \text{Col A} = 2$.

To find $\text{Nul}\, A$ we solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = 0 \end{cases} \text{ and}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ so } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Nul}\, A \text{ and } \dim \text{Nul}\, A = 2.$$

$\dim \text{Col}\, A$ = number of pivot columns of A, $\dim \text{Nul}\, A$ = number of free variables of A
We see that for an $m \times n$ matrix, we an write the following equation in three different ways:

$$\text{number of pivots} + \text{number of free variables} = \text{total number of variables.}$$

$$\dim\ \text{Col}\ A + \dim\ \text{Nul}\ A = \#\ \text{columns}\ .$$

$$\text{rank} \quad + \quad \text{nullity} = n.$$

This is the **rank theorem**.

4.6 Rank.

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A, written $\text{Row}\, A$.

Ex 1 Let $A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix}$ and $\begin{aligned} \mathbf{r}_1 &= (-1, 2, 3, 6) \\ \mathbf{r}_2 &= (2, -5, -6, -12) \\ \mathbf{r}_3 &= (1, -3, -3, -6) \end{aligned}$

$\text{Row}\, A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a subspace of $\mathbf{R}^4$.
We can also express row vectors horizontally and we conclude that $\text{Col}\, A^T = \text{Row}\, A$.

When we do row operations to reduce matrix A to matrix B we take linear combinations of the rows of A to come up with B. This process can be reversed to get back from B to A. Because of this we have:
Th If two matrices A and B are row equivalent then their row spaces are the same. If B is in echelon form, the nonzero rows form a basis for the row space of A and B.

Ex 2 Find a basis for the row space, column space and null space of A in Ex 1, and state the dimensions of each.

Sol $A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$

A basis for $\text{Row}\, A$ is $\{(-1, 2, 3, 6), (0, -1, 0, 0)\}$, so $\dim \text{Row}\, A = 2$.

Basis for $\text{Col}\, A$: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \right\}$, since the first and second columns of B and hence of A are the pivot columns. Hence $\dim \text{Col}\, A = 2$.

To find $\text{Nul}\, A$, solve $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3+6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ basis for Nul}\, A\text{: } \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \ \dim \text{Nul}\, A = 2.$$

rank $A = \dim \text{Col}\, A =$ number of pivot columns of $A = \dim \text{Row}\, A$.
nullity $A = \dim \text{Nul}\, A =$ number of nonpivot columns of A.

Since the number of pivot columns plus the number of nonpivot columns is equal to the total number of columns we have proven:
The Rank Theorem If A is an $m \times n$ matrix then rank $A +$ nullity $A = n$.

Ex Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$. One can easily check that

a basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and a basis for Row $A = \{[\,1 \quad 0 \quad -1\,]\}$
Hence Nul A is a plane and Row A is a line perpendicular to the plane.