

Lecture 13.

From last time: For an $m \times n$ matrix A :

$$\text{Rank of } A = \dim \text{Col } A = \dim \text{Row } A = \#\text{pivots.}$$

$$\text{Nullity of } A = \dim \text{Nul } A = \#\text{free variables.}$$

$$\text{number of pivots} + \text{number of free variables} = \text{total number of variables.}$$

$$\text{rank} + \text{nullity} = n.$$

The row space of A is perpendicular to the null space of A . **Ex** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$.

One can easily check that

$$\text{a basis for Nul } A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and a basis for Row } A = \{[1 \ 0 \ -1]\}$$

Hence $\text{Nul } A$ is a plane and $\text{Row } A$ is a line perpendicular to the plane.

4.4 Coordinate Systems: Recall that $\mathbf{b}_1, \dots, \mathbf{b}_n$ form a **basis** for a vector space V if

- (i) $\mathbf{b}_1, \dots, \mathbf{b}_n$ are linearly independent, and
- (ii) $\mathbf{b}_1, \dots, \mathbf{b}_n$ span V .

The Unique Representation Theorem Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be a basis for a vector space V . Then for any $\mathbf{x} \in V$, there is unique set of scalars c_1, \dots, c_n such that

$$(4.1) \quad \mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

Pf Since $\mathbf{b}_1, \dots, \mathbf{b}_n$ span V there are scalars such that (4.1) holds.

If there was another set of scalars d_1, \dots, d_n such that $\mathbf{x} = d_1 \mathbf{b}_1 + \dots + d_n \mathbf{b}_n$ then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1) \mathbf{b}_1 + \dots + (c_n - d_n) \mathbf{b}_n$$

and since $\mathbf{b}_1, \dots, \mathbf{b}_n$ are linearly independent it follows that $c_1 - d_1 = \dots = c_n - d_n = 0$.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis. The **\mathcal{B} -coordinates** of \mathbf{x} are the weights c_1, \dots, c_n such that $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n$. The vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **\mathcal{B} -coordinate vector** of \mathbf{x} . The mapping $\mathbf{x} \rightarrow [\mathbf{x}]_{\mathcal{B}}$ is called the **\mathcal{B} -coordinate mapping**.

Ex Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{E}}$.

Sol We can write

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. We want to find c_1 and c_2 such that

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solving the system gives that $c_1 = 2$ and $c_2 = 3$. Hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

One can graph this in standard \mathcal{E} graph paper and in \mathcal{B} graph paper.

Note that in the example

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

In general for a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$;

$$\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}, \quad \text{where} \quad P_{\mathcal{B}} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$$

and $[\mathbf{x}]_{\mathcal{B}}$ the coordinate vector. We call $P_{\mathcal{B}}$ the **change-of-coordinate matrix** from the standard basis in \mathbf{R}^n to the basis \mathcal{B} .

Ex Find the coordinates of $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ in the basis $\mathbf{b}_1, \mathbf{b}_2$ in the previous example.

Sol $P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2] = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ and $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix}$. Then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \mathbf{x} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

The coordinate mapping allows us to introduce coordinate systems for unfamiliar vector spaces:

Ex Standard basis for the polynomials of degree 2 or less, \mathbf{P}_2 is $\{1, t, t^2\}$. We can

write $[a + bt + ct^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. We say that \mathbf{P}_2 is isomorphic to \mathbf{R}^3 . All vector space

operations in \mathbf{P}_2 corresponds to operations in \mathbf{R}^3 , e.g. adding two polynomials

$(-1 + 2t - 3t^2) + (2 + 3t + 5t^2) = 1 + 5t + 2t^2$ corresponds to adding their coordinate vectors: $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$.

Th If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V then the coordinate map $\mathbf{x} \rightarrow [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V to \mathbf{R}^n .

Assume that \mathcal{B} is a basis. Show that the set $\mathbf{u}_1, \dots, \mathbf{u}_p$ are linearly independent if and only if $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$ are linearly independent.

Coordinate vectors allow us to associate vector spaces with subspaces:

Ex Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$. Then $\mathbf{x} =$

$\begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} \in H$. Find $[\mathbf{x}]_{\mathcal{B}}$.

Sol Find c_1 and c_2 such that $c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$. Augmented matrix

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

so $c_1 = 3$ and $c_2 = 4$ and hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Every vector on H is associated with a unique vector in \mathbf{R}^2 . H is isomorphic to \mathbf{R}^2 .

Draw the diagram of the net of $\mathbf{b}_1, \mathbf{b}_2$ on H .