Lecture 16: 3.2 Properties of determinants.

Recall: Let $A=(a_{ij})$ be an $n\times n$ matrix and let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The **cofactors** are

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

and the **determinant** is

$$\det A = a_{11}C_{11} + ... + a_{n1}C_{n1}. \tag{2}$$

We saw that if E is an elementary matrix then

$$(*) \quad \det(EA) = \begin{cases} -\det(A), & \text{if } E \text{ is of type I: interchange two rows.} \\ \alpha \det(A), & \text{if } E \text{ is of type II: multiply row by } \alpha. \\ \det(A), & \text{if } E \text{ is of type III: add multiple of row to another.} \end{cases}$$

Example 1. Find the determinant of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution. Adding -2 times the 1st row to the 2nd and expanding along the first column:

$$\begin{vmatrix} 1 & 5 & 1 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 1 \\ 0 & -6 & -3 \\ 0 & -2 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -6 & -3 \\ -2 & 0 \end{vmatrix} = 1 \cdot \big((-6)0 - (-3)(-2)\big) = -6.$$

By letting E in (*) act on the identity I in place of A, using that $\det(I) = 1$, we get

$$\det(E) = \begin{cases} -1, & \text{if } E \text{ is of type I} \\ \alpha, & \text{if } E \text{ is of type II} \\ 1, & \text{if } E \text{ is of type III} \end{cases}$$

It therefore follows that if E is an elementary matrix

$$\det(EA) = \det(E)\det(A) \tag{3.2.1}$$

Theorem. $\det(AB) = \det(A)\det(B)$.

Proof. If A is nonsingular it can be written as a product of elementary matrices $E_k \cdots E_1$:

$$\det(AB) = \det(E_k \cdots E_1 B) = \det(E_k) \cdots \det(E_1)\det(B) = \det(A)\det(B).$$

Theorem. $\det A^T = \det A$.

Proof. This is clear for the elementary matrices. Indeed, for type I and II, for example

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{or} \qquad E = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have $E^T = E$ and so $\det E^T = \det E$. The transpose of a type III elementary matrix is also of type III, and both therefore have determinant equal to 1. Now for a general matrix A, either A and A^T are both singular and hence both have determinant zero, or we can write A as a product of elementary matrices $E_k \cdots E_1$, and

$$\begin{aligned}\det A^T &= \det(E_k \cdots E_1)^T = \det(E_1^T \cdots E_k^T) = \det E_1^T \cdots \det E_k^T \\ &= \det E_1 \cdots \det E_k = \det(E_1 \dots E_k) = \det A.\end{aligned}$$

Theorem. For any i and j we have

$$\det A = a_{i1}C_{i1} + ... + a_{in}C_{in} = a_{1j}C_{1j} + ... + a_{nj}C_{nj}$$

Proof. We can show by induction on n that

$$\det A = a_{11}C_{11} + \dots + a_{1n}C_{1n}.$$

For example, we do this in three dimensions:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then

$$\det A = \det A^T = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}.$$

In order to show that

$$\det A = a_{21}C_{21} + a_{22}C_{22} + \dots + a_{2n}C_{2n}$$

we use the fact that if E is the elementary matrix which interchanges the first two rows, then $\det EA = -\det A$. We carry this out in 3 dimensions. Indeed

$$\det A = -\det(EA) = -\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

$$= -a_{31}\begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} + a_{32}\begin{vmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{vmatrix} - a_{33}\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$$

$$= (-1)^{3+1}a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2}a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3}a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 2.

$$\begin{vmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-2)2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (-2)2(1 \cdot 1 - 1 \cdot 0) = -4$$

3.3 Determinants and volumes.

Th If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.
Pf of the 2×2 cases. The theorem is obviously true for diagonal 2×2 matrices:

$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \text{Area of rectangle with sides } a \text{ and } d$$

We will show that any 2×2 matrix $A = [\,\mathbf{a}_1\ \mathbf{a}_2\,]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$. We know that the absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another and its easy to see that one can transform A into diagonal form with such operations. Column interchanges do not change the parallelogram at all so it suffices to prove the following fact: The area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$ for any c. This follows from that the points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to the line through $\mathbf{0}$ and $\mathbf{a}_1$.