

Lecture 17: Dynamical systems and eigenvectors.

Example 1. In a certain town, 30% of the married men get divorced each year and 20% of the single men get married each year. Suppose that initially there are 8000 married men and 2000 single men. What is the proportion married as $k \rightarrow \infty$?

Solution. Let

$$\mathbf{w}_k = \begin{bmatrix} w_{k1} \\ w_{k2} \end{bmatrix} = \begin{bmatrix} \text{number of married men after } k \text{ years} \\ \text{number of single men after } k \text{ years} \end{bmatrix}.$$

Let A be the 2×2 matrix such that

$$\mathbf{w}_{k+1} = A\mathbf{w}_k,$$

$$A = \begin{bmatrix} \text{proportion of the married} & \text{proportion of the singles} \\ \text{who stay married that year} & \text{who get married that year} \\ \text{proportion of married} & \text{proportion of the singles} \\ \text{who get divorced that year} & \text{who get married that year} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

$$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}. \text{ After the first year we get } \mathbf{w}_1 = A\mathbf{w}_0 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}.$$

After the second year we get $\mathbf{w}_2 = A\mathbf{w}_1 = A^2\mathbf{w}_0$ and so on:

$$\mathbf{w}_k = A^k \mathbf{w}_0$$

$$\text{It seems like as } k \rightarrow \infty, \mathbf{w}_k \text{ converges: } \mathbf{w}_{10} = \begin{bmatrix} 4004 \\ 5996 \end{bmatrix}, \mathbf{w}_{20} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}, \mathbf{w}_{30} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}.$$

In fact, any initial condition will converge to the **steady state** $(4000, 6000)^T$, for which the number of divorces $0.3 \cdot 4000$ is equal to the number of marriages $0.2 \cdot 6000$.

If we start with $\mathbf{x}_1 = (2, 3)^T$ proportional to the steady state we get back \mathbf{x}_1 :

$$A\mathbf{x}_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{x}_1$$

There is another vector $\mathbf{x}_2 = (-1, 1)^T$ that A acts on by simply multiplying by $1/2$:

$$A\mathbf{x}_2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

The vectors $\mathbf{x}_1, \mathbf{x}_2$ form a basis so we can write our initial condition in terms of these:

$$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = 2000 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 4000 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2000\mathbf{x}_1 - 4000\mathbf{x}_2$$

Then

$$\mathbf{w}_k = A^k \mathbf{w}_0 = 2000A^k \mathbf{x}_1 - 4000A^k \mathbf{x}_2 = 2000\mathbf{x}_1 - 4000 \frac{1}{2^k} \mathbf{x}_2$$

and $\mathbf{w}_k \rightarrow 2000\mathbf{x}_1 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$, as $k \rightarrow \infty$.

A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and a corresponding vector \mathbf{x} is called an **eigenvector**.

We just calculated $A^k \mathbf{x}$ for large k using the eigenvalues and eigenvectors.

We express $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ in terms of the basis of eigenvectors $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2$.

Change of coordinates $\mathbf{x} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = [\mathbf{x}_1 \ \mathbf{x}_2]$, so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$. Then

$$A^k \mathbf{x} = c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD^k P^{-1} \mathbf{x}, \text{ where } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Hence $A = PDP^{-1}$ and $A^k = (PDP^{-1})^k = PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^k P^{-1}$.

Example 2. Consider the matrix $A = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$.

Find the eigenvectors and eigenvalues and describe the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ geometrically.

Solution. Solving

$$A\mathbf{x} = \lambda\mathbf{x}$$

Is equivalent to solving

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

There is a non-trivial solution exactly when $A - \lambda I$ is not invertible, that is

$$0 = \begin{vmatrix} \frac{9}{10} - \lambda & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} - \lambda \end{vmatrix} = \frac{1}{100} \begin{vmatrix} 9 - 10\lambda & 3 \\ 3 & 1 - 10\lambda \end{vmatrix}$$

Hence

$$0 = (9 - 10\lambda)(1 - 10\lambda) - 9 = 100\lambda^2 - 100\lambda = 100\lambda(\lambda - 1).$$

Hence $\lambda = 1$ or $\lambda = 0$. In the first case with eigenvalue 1, we get the eigenvector

$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $A\mathbf{x}_1 = \mathbf{x}_1$. In the second case with eigenvalue 0, we get the eigen-

vector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $A\mathbf{x}_2 = \mathbf{0} = 0 \cdot \mathbf{x}_2$. If we express If $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, in terms of

the basis of eigenvectors then $A\mathbf{x} = c_1\mathbf{x}_1$. We see that if L be the line in \mathbf{R}^2 that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then T be the linear transformation that projects any vector orthogonally onto L .

Change of coordinates $\mathbf{x} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = [\mathbf{x}_1 \ \mathbf{x}_2] = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$, where $P^{-1} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$.

Hence $A\mathbf{x} = c_1\mathbf{x}_1 = [\mathbf{x}_1 \ \mathbf{x}_2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

The matrix D for T in the $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2\}$ coordinate system is hence very simple. The matrix for A for T in the standard coordinates is more complicated. The following diagram commute

$$\begin{array}{ccc} c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{x} & \xrightarrow{A} & A\mathbf{x} = c_1\mathbf{x}_1 \\ \uparrow P & & \uparrow P \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{D} & [A\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \end{array},$$

Ex 3 Let T be the linear transformation rotating a vector an angle θ . The matrix

for T is $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Find the eigenvectors and eigenvalues of T .

Sol Unless θ is a multiple of π it does not have any real eigenvalues and eigenvectors. If θ is a multiply of π the eigenvalues are ± 1 .