MATH 20F LINEAR ALGEBRA

Lecture 4: 1.4 Matrix Multiplication and the Matrix Equation.

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{in} \\ & \vdots & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Define the product $A\mathbf{x} \in \mathbb{R}^m$ by

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_m$$

Then **Matrix multiplication** the matrix A defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Notice that

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{bmatrix}.$$

The jth term down is obtained by dotting the jth row of A with $\mathbf{x}$.

Ex $\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}.$

Alternatively $\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + (-4) \cdot (-6) \\ 3 \cdot 7 + 2 \cdot (-6) \\ 0 \cdot 7 + 5 \cdot (-6) \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}.$

We can write the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1.4.1}$$

as a single matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

The map is linear; $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$. All linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are of this form.

We can now write a linear system with augmented matrix

$$(1.4.5)\qquad \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix},$$

as a **System of Linear Equations**

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 + x_3 &= -1 \end{aligned}$$

as a **Vector Equation**

$$(1.4.7)\qquad x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

or as a **Matrix Equation**

$$(1.4.8)\qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Viewing the system as the intersection of planes (1.4.6) is called the **row picture** since each equation corresponds to a row of the augmented matrix (1.4.5).
Viewing the system as a linear combination of vectors (1.4.7) is called the **column picture** since each vector corresponds to a column of the augmented matrix (1.4.5).
In (1.4.8) we think of the system as a map taking $\mathbf{R}^3 \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^2$.
For large systems it is much easier to think in the column picture.

Ex Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$. For which $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is the equation $A\mathbf{x} = \mathbf{b}$ consistent?
Sol The augmented matrix corresponding to the system is

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{matrix} \\ +3(1) \\ -2(1) \end{matrix} \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{bmatrix}$$

Hence $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_3 - 2b_1 = 0$, which is a plane in $\mathbf{R}^3$. Hence if $\mathbf{a}_j$ denotes the columns of A, then $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ for some $\mathbf{x}$ if and only if $b_3 - 2b_1 = 0$. Therefore the columns of A span the plane $b_3 - 2b_1 = 0$.

Let A be an $m \times n$ matrix. We say that the columns of $A = [\,\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_p\,]$ **span** $\mathbf{R}^m$ if every vector $\mathbf{b} \in \mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_p$, i.e. $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_p$.
Th The following statements are equivalent.
(a) For each $\mathbf{b} \in \mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
(b) Each $\mathbf{b} \in \mathbf{R}^m$ is a linear combination of the columns of A.
(c) The columns of A span $\mathbf{R}^m$.
(d) A has pivot position in every row.
Pf That (a), (b), (c) are equivalent follows directly from the definitions. It therefore

suffices to show that (a) is true when (d) is true and that (a) is false when (d) is false. Suppose (d) is true. If we reduce the augmented matrix to row echelon form $[\,A\ \ \mathbf{b}\,] \sim [\,U\ \ \mathbf{d}\,]$, Since each row of U has a pivot position there is no pivot position in the last column of $[\,U\ \ \mathbf{d}\,]$, so the equation $A\mathbf{x} = \mathbf{b}$ has a solution, i.e. (a) is true. Suppose (d) is false. Then the last row of the row-echelon form $U \sim A$ contains all zeros. Let $\mathbf{d}$ be a vector with 1 as last entry. If we reverse all the row operations $[\,U\ \mathbf{d}\,] \sim [\,A\ \mathbf{b}\,]$ we get a system $A\mathbf{x} = \mathbf{b}$ which is inconsistent so (a) is false.

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

Sol No, since A has two columns it has at most two pivots and none in the last row.

Ex Let $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ Is $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

Sol Since $\mathbf{v}_2 = 2\mathbf{v}_1$ we have

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = (x_1 + 2x_2)\mathbf{v}_1$$

so all linear combinations are on the line in the direction of $\mathbf{v}_1$ so $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line.

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $\mathbf{b}$ in the plane spanned by the columns of A?

Sol The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

The system is inconsistent so $\mathbf{b}$ is not in the plane spanned by the columns of A.