Lecture 5: 1.5 Solution sets of linear systems.
A **homogeneous system** is a system of the form

$$A\mathbf{x} = \mathbf{0}$$

where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbf{R}^m$.
A homogeneous system always has the **trivial solution** $\mathbf{x} = \mathbf{0}$ so its consistent.
Consistent systems with a free variable have infinitely many solutions.
A homogeneous system has a **nontrivial solution** if and only if it has free variables.

Ex 1 Determine if the following system has nontrivial solution set and describe it.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 + 8x_2 - 11x_3 &= 0 \end{aligned}$$

Sol There is at least one free variable since there are 2 equations in 3 variables.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -11 & 0 \end{bmatrix} \sim \begin{matrix} \\ -4(1) \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{matrix} +3(1) \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad x_1 = -2x_2, \quad x_2 = \text{free}, \quad x_3 = 0$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \mathbf{v}$$

This is the parametric equation of a line through $\mathbf{0}$ in the direction of $\mathbf{v}$
Ex 2 Determine the solution set of

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 + 8x_2 - 11x_3 &= 2 \end{aligned}$$

Sol $$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -11 & 2 \end{bmatrix} \sim \begin{matrix} \\ -4(1) \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{matrix} +3(1) \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}, \quad x_2 \text{ is a free parameter.}$$

This is the parametric equation of a line through $\mathbf{p}$ in the direction of $\mathbf{v}$.
The line in Ex 2 is parallel to the line in Ex 1.

If the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is consistent its solution set is parallel to the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Th Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and let $\mathbf{p}$ be a solution. Then any other solution $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is a solution to $A\mathbf{v}_h = \mathbf{0}$.
Pf Since matrix multiplication is linear $A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b}$.

Ex Describe the solution set to $x_1 - 2x_2 - 2x_3 = b$ for $b = 0, 1$.
Sol x_2 and x_3 are free variables

$$\mathbf{x} = \begin{bmatrix} b + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}_1 + x_3 \mathbf{v}_2$$

Since x_2 and x_3 are free parameters this is the **parametric vector equation** of a plane through $\mathbf{p}$ and parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
If $b=0$ its the plane spanned by $\mathbf{v}_1, \mathbf{v}_2$ and if $b \neq 0$ it is a plane parallel to this plane.

1.7 Linear Independence. The homogeneous equations $A\mathbf{x} = \mathbf{0}$ can be studied from a different perspective by writing them as vector equations;

$$(1.7.1)\quad \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there are weights $\lambda_1, \dots, \lambda_p$ not all 0, such that

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_p\mathbf{v}_p = \mathbf{0}.$$

Ex Determine if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$ are linearly independent?

Sol They are linearly dependent if (1.7.1) has a nontrivial solution. Augmented:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free there are nontrivial solutions $x_1 = -33x_3$, $x_2 = 18x_3$, x_3 is free. If we e.g. let $x_3 = 1$ then $x_1 = -33$ and $x_2 = 18$ so we have the linear dependence relation

$$-33 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The columns of A are linearly independent $\Leftrightarrow$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Ex Determine if $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are linearly dependent.

Sol Note that $\mathbf{u}_2 = 2\mathbf{u}_1$ so $2\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}$, i.e. they are linearly dependent.

Ex Determine if $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are linearly dependent.

Sol Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. If say $c_1 \neq 0$ then $\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2$ which is impossible. Hence $c_1 = 0$ and similarly $c_2 = 0$ so $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent.

Two vectors are linearly dependent if and only if one is a multiple of the other.

Th p vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ are linearly dependent if $p > n$.

Pf $A = [\,\mathbf{v}_1 \cdots \mathbf{v}_p\,]$ is a $p \times n$ matrix. If $p > n$ then $A\mathbf{x} = \mathbf{0}$ has more variables than equations and there are free variables. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This is the same as that the columns of A are linearly dependent.

Ex With as little work as possible decide if the following sets of vectors are linearly dependent. (a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$ and (b) the columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \end{bmatrix}$.

Sol (a) linearly independent since one is not a multiple of the other. (b) linearly dependent since there are more columns than entries in the vectors.

Th If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly dependent then one is a linear combination of the others.