Lecture 8: 2.2 The inverse of a matrix.

Definition: The identity matrix is $I = [\,\delta_{ij}\,]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{in case } 4 \times 4.$$

Definition: The transpose A^T is the matrix with rows and columns interchanged, $(A^T)_{ij} = (A)_{ji}$

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$.

Which is true?

$$(AB)^T = A^T B^T \text{ or } (AB)^T = B^T A^T?$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I \tag{2.2.1}$$

where I is the identity matrix. The matrix A^{-1} called the **inverse** of A is unique. In fact if $BA = I$ then $B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}$.
Not all $n \times n$ matrices are invertible. A matrix which is not invertible is called **singular**. An invertible matrix is called **nonsingular**.

An **inverse** of a transformation $\mathbf{x} \to T(\mathbf{x})$ is a transformation which takes you back $T(\mathbf{x}) \to \mathbf{x}$. The condition $A^{-1}A = I$ says that the inverse of the linear transformation $\mathbf{x} \to A\mathbf{x}$ is the linear transformation $\mathbf{y} \to A^{-1}\mathbf{y}$. In fact, if we compose $\mathbf{x} \to A\mathbf{x}$ with $\mathbf{y} \to A^{-1}\mathbf{y}$ we get $\mathbf{x} \to A\mathbf{x} \to A^{-1}(A\mathbf{x}) = (AA^{-1})\mathbf{x} = I\mathbf{x} = \mathbf{x}$.

Question: For 2×2 matrices, what is the inverse of a scaling by a factor 3 and what is its matrix?
What is the inverse of a rotation counterclockwise angle $\pi/2$ and what is its matrix?

Th If A is invertible then the equations $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
Pf If A is invertible then

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\,\mathbf{b} = \mathbf{b},$$

so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution. To see that it is unique, suppose we have some solution $A\mathbf{x} = \mathbf{b}$ and multiply both sides by A^{-1} to get

$$\mathbf{x} \;=\; (A^{-1}A)\mathbf{x} \;=\; A^{-1}(A\mathbf{x}) \;=\; A^{-1}\mathbf{b}.$$

Th Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible and $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad - bc = 0$ then A is not invertible.
Pf If $ad - bc \neq 0$ its easy to check that $AA^{-1} = A^{-1}A = I$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}.$$

If $ad - bc = 0$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence either $a = b = c = d = 0$, or we have a non-zero vector $\mathbf{v}$ with $A\mathbf{v} = \mathbf{0}$ and A is not invertible.
Ex Solve the system

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}, \ A^{-1} = \frac{1}{7 \cdot 2 - 3 \cdot 5} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$ as is clear from the composition of transformations.

Question: How can we calculate the inverse of an $n \times n$ matrix?
Recall: We solve $A\mathbf{x} = \mathbf{b}$ by row reduction to reduced echelon form. If A is invertible, then we can solve this equation uniquely for each $\mathbf{b}$ and so the number of pivots is n. The solution is given by

$$A\mathbf{v} = \mathbf{b} \qquad \Leftrightarrow \qquad [A, \mathbf{b}] \sim [I, \mathbf{v}].$$

Writing

$$A^{-1} = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3],$$

We have

$$AA^{-1} = I \qquad \Rightarrow A\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\left[A, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \sim [I, \mathbf{v}_1], \qquad \left[A, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right] \sim [I, \mathbf{v}_2], \qquad \left[A, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \sim [I, \mathbf{v}_3].$$

Putting these together, we have

$$[A, I] \sim [I, A^{-1}].$$

Ex Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim (2) + 3(1) \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (3) \\ (2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{bmatrix}$$

The inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

To compute the inverse of the $n \times n$ matrix A: Calculate the reduced row echelon form of the augmented matrix $[\,A \;\; I\,]$. If it is of the form $[\,I \;\; B\,]$ then A is invertible and $A^{-1} = B$. Otherwise A is not invertible.

One can also prove that this works multiplying by **elementary** matrices which correspond to elementary row operations. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Multiplying by E_1 adds 3 times row one to row two:

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiplying by E_2 switches row two and row three:

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence

$$E_2 E_1 A = I$$

and multiplying both sides by A^{-1} to the right gives since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$:

$$E_2 E_1 I = A^{-1}$$

Hence a sequence of elementary row operations that reduce A to I reduce I to A^{-1}.

2.3 Characterizations of Invertible Matrices.

Question: What conditions are equivalent to that a matrix is invertible?

Invertible Matrix Theorem (IMT)

Let A be a given $n \times n$ matrix. Then the following are equivalent:

a) A is invertible.
b) A is row equivalent to I.
c) A has n pivot positions
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A are linearly independent.
f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
g) The equations $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.
h) The columns of A span $\mathbf{R}^n$.
i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is onto.
j) There is an $n \times n$ matrix C such that $CA = I$.
k) There is an $n \times n$ matrix D such that $AD = I$.
l) A^T is invertible.