Math 20F Solutions for Practice Midterm I Spring 06, Lindblad.

1. (a) Determine if the following systems are consistent and if so give all solutions in parametric form:

$$(a)\quad \begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - 7x_2 &= 0 \\ -5x_1 + 8x_2 &= 5 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned}$$

Sol (a) Row reduction gives

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{array}\right]$$

The system is inconsistent since writing out the system in terms of x_1 and x_2 again the last equation says that $0 = -12$.
Sol (b) Row reduction gives

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \Leftrightarrow$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 2 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

The system can now be solved. $x_4 = \alpha$ is a free variable $(x_1, x_2, x_3, x_4) = (2 - 6\alpha, 4 + \alpha, 3 - \alpha, \alpha)$.

2. (a) Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$ (b) Find the solution of $A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Sol (a)
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow \begin{array}{c} \\ II - 2I \\ III - 3I \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right)$$

$$\Leftrightarrow \begin{array}{c} \\ \\ III - 5II \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right) \Leftrightarrow \begin{array}{c} \\ \\ -III \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right)$$

$$\Leftrightarrow \begin{array}{c} I - III \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right) \Leftrightarrow \begin{array}{c} I - II \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right)$$

(a) $A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$ and (b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -3 \end{bmatrix}$.

3. A linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ satisfies $T(\mathbf{e}_1) = (2,1,1)$ and $T(\mathbf{e}_2) = (1,1,2)$, where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$.
(a) Find the standard matrix for T.
(b) Find $T((-3,4))$.
(c) Is T one-to-one? Justify your answer.
(d) Is T onto? Justify your answer.

Sol (a) $A = [\,T(\mathbf{e}_1), T(\mathbf{e}_2)\,] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) $T((-3,4)) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$.

(c) T is one-to-one since the columns of A are linearly independent.
(d) T is not onto since the columns of A can not span $\mathbf{R}^3$ since its only two vectors.

4.(a) Show that $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) Find a basis for Nul A.
(c) Find a basis for Col A.

Sol(a) $$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $A\mathbf{x} = \mathbf{0}$ is equivalent to $B\mathbf{x} = \mathbf{0}$ and hence
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_3 + 3x_5 \\ 3x_3 - 5x_5 \\ x_3 \\ 4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$
These two vectors form a basis for Nul A.

(c) Since the first, second and fourth column of B are pivot columns the corresponding columns of A form a basis for Col A, i.e.
$$\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix}$$ form a basis for Col A.