Solutions to Math 21C Final, Winter 02. Agler.

1. Two vectors for two of the sides of the triangle are $\mathbf{a} = \langle 1,2,1\rangle$ and $\mathbf{b} = \langle 2,-2,3\rangle$. The area of the triangle is $|\mathbf{a}\times\mathbf{b}|/2$. We have $\mathbf{a}\times\mathbf{b} = ... = 8\,\mathbf{i} - \mathbf{j} - 6\,\mathbf{k}$ and $|\mathbf{a}\times\mathbf{b}| = \sqrt{8^2+1+6^2} = \sqrt{101}$.

2. The curves intersect when $\langle t,t^2,t^3\rangle = \langle 1+s,4s,8s^2\rangle$ for some t and s. This means that $t = 1+s$ and $t^2 = 4s = 4(t-1)$ so $t^2-4t+4 = 0$ and hence $(t-2)^2 = 0$, i.e. $t = 2$ and hence $s = 1$. Its is easy to check that also the last equation is satisfied for these values. We have $\mathbf{r}_1'(t) = \langle 1,2t,3t^2\rangle$ so $\mathbf{r}_1'(2) = \langle 1,4,12\rangle$ and $\mathbf{r}_2'(s) = \langle 1,4,16s\rangle$ so $\mathbf{r}_2'(1) = \langle 1,4,16\rangle$. The angle is given by $\cos\theta = \mathbf{r}_1'(2)\cdot\mathbf{r}_2'(1)/(|\mathbf{r}_1'(2)|\,|\mathbf{r}_2'(1)|) = 209/(\sqrt{1+4^2+12^2}\cdot\sqrt{1+4^2+16^2}) = 209/\sqrt{161\cdot 273}$.

3. The vector $\mathbf{n} = \langle 2,1,4\rangle$ between the points is normal to the plane and the point in the middle between the points $P = (0,3/2,1)$ is on the plane. The equation of the plane is therefore $2(x-0)+1(y-3/2)+4(z-1) = 0$.

4. $\mathbf{r}'(t) = 3t^{1/2}\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. The initial speed is $|\mathbf{r}'(0)| = |2\mathbf{k}| = 2$. The arc length is $\int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \sqrt{9t+4}\,dt = (9t+4)^{3/2}2/27\big|_0^1 = 2(13^{3/2}-4^{3/2})/27$.

5. The tangent plane to $F(x,y,z) = x^2+y^2/4+z^2/9 = 1$ at a point (x_0,y_0,z_0) has normal $\nabla F(x_0,y_0,z_0) = \langle 2x_0, y_0/2, 2z_0/9\rangle$. This vector is parallel to the normal of the plane $x+y-z=0$, which is $\langle 1,1,-1\rangle$, if $2x_0=\lambda$, $y_0/2=\lambda$ and $2z_0/9=-\lambda$ Since the point also must lie on the surface we must have $F(\lambda/2,2\lambda,-9\lambda/2) = \lambda^2/4+\lambda^2+9\lambda^2/4 = 7\lambda^2/2 = 1$ so $\lambda = \pm\sqrt{2/7}$. The point is $(x_0,y_0,z_0) = \pm\sqrt{2/7}(1/2,2,-9/2)$.

6. $f_x(x,y) = 4x^3-8y = 0$ and $f_y(x,y) = -8x+4y = 0$ gives $y = 2x$ and $4x(x^2-4) = 0$. Hence $x=0$ or $x=\pm 2$ so the critical points are $(0,0), (2,4), (-2,-4)$. $f_{xx} = 12x^2$, $f_{xy} = -8$, $f_{yy} = 4$ so $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 48x^2-64$. Then $D(0,0) < 0$ so $(0,0)$ is a saddle point, $D(2,4) = 128 > 0$ and $f_{xx}(2,4) = 48 > 0$ so $(2,4)$ is local min, $D(-2,-4) = 128 > 0$ and $f_{xx}(-2,-4) = 48 > 0$ so $(-2,-4)$ is local min.

7. Minimize the area $A = xy+2xz+2yz$, subject to the constraint that the volume is $V = xyz = 32,000$. Lagrange multiplies: $\nabla A(x,y,z) = \langle y+2z, x+2z, 2x+2y\rangle$ and $\nabla V(x,y,z) = \langle yz, xz, xy\rangle$ so we must find all (x,y,z) and λ such that $y+2z = \lambda yz$, $x+2z = \lambda xz$, $2x+2y = \lambda xy$ and $V(x,y,z) = 32,000$. Multiplying the first equation by x, the second by y and the third by z we get $x(y+2z) = y(x+2z) = z(2x+2y)$. Subtracting the first two equations gives $2z(x-y) = 0$ and subtracting the first and third gives $(x-2z)y = 0$. If $z = 0$ then $y = 0$ or $x = 0$. If $z \neq 0$ then $x = y = 0$ or $x = y = 2z$. Hence we have the points $(x,0,0)$, $(0,y,0)$, $(0,0,z)$ and $(2z,2z,z)$. Only the last one gives $V \neq 0$ and we must have $V(2z,2z,z) = 4z^3 = 32,000$ which is equivalent to $z = 20$ so $(x,y,z) = (40,40,20)$.

8. Let $D = \{(x,y);\ 10-3x^2-3y^2 \geq 4\} = \{(x,y);\ x^2+y^2 \leq 2\}$.
The volume is $\iint_D z\,dA = \iint_D (10-3x^2-3y^2)\,dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (10-3r^2)\,r dr\,d\theta = \int_0^{2\pi} (5r^2-3r^4/4)\big|_0^{\sqrt{2}}\,d\theta = \int_0^{2\pi} 7\,d\theta = 14\pi$.

9. $\iint_T \sqrt{1+1+4y^2}\,dA = \int_0^1\int_0^y \sqrt{2+4y^2}\,dx\,dy = \int_0^1 \sqrt{2+4y^2}x\big|_{x=0}^{y}\,dy = \int_0^1 \sqrt{2+4y^2}y\,dy (2+4y^2)^{3/2}/12\big|_0^1 = (6^{3/2}-2^{3/2})/12$.

10. $E = \{(x,y,z);\ x \geq 0, y \geq 0, z \geq 0, x+y+z/2 \leq 1\}$ $= \{(x,y,z);\ 0 \leq z \leq 2,\ 0 \leq y \leq 1-z/2,\ 0 \leq x \leq 1-y-z/2\}$.
$\iiint_E y \quad dV = \int_0^2\int_0^{1-z/2}\int_0^{1-y-z/2} y\,dx\,dy\,dz = \int_0^2\int_0^{1-z/2} yx\big|_{x=0}^{1-y-z/2}\,dy\,dz$ $= \int_0^2\int_0^{1-z/2} y(1-y-z/2)\,dy\,dz = \int_0^2 (y^2/2-y^3/3-zy^2/4)\big|_{y=0}^{1-z/2}\,dz$ $= \int_0^2 (1-z/2)^3/6\,dz = \int_{1/2}^1 t^3/3\,dt = t^4/12\big|_0^1 = 1/12$.