Lecture 1: Describing curves in the plane.

A curve in the plane can be defined by:

A graph

$$y = F(x), \qquad a \leq x \leq b$$

(or $x = G(y)$).

A Cartesian equation

$$h(x, y) = 0$$

Parametric equations

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases}, \qquad \text{where} \qquad \alpha \leq t \leq \beta.$$

Think of t as the time and (x, y) as the position of particle at time t.
The particle moves and traces out a curve.

Example. The curve $\begin{cases} x = \cos t, \\ y = \sin t \end{cases}$, where $0 \leq t \leq 2\pi$ is a parametrization of the circle, since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. The circle is not a graph. It is however a union of the two graphs, $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ where $-1 \leq x \leq 1$.

Example. The curve $\begin{cases} x = \cos 2t, \\ y = \sin 2t \end{cases}$, where $0 \leq t \leq \pi$ is another parametrization of the circle, since $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$.

Note that both examples above describe the same curve. The difference is that the particle travels twice as fast in the second example.

Example. Write the curve given in parametric form by $\begin{cases} x = t^2, \\ y = t^3 \end{cases}$ as a graph and as a parametric equation.

When we try to write it as a graph $y = F(x)$ we encounter a problem. Notice that for points on the curve, $x \geq 0$. Writing t in terms of x, if $t \geq 0$ we have $t = x^{1/2}$, while if $t < 0$ we have $t = -x^{1/2}$, so there are two possible values of t for every $x > 0$. Plugging into the equation for y, we get two graphs,

$$\text{(I)} \quad y = x^{3/2}, \qquad\qquad \text{(II)} \quad y = -x^{3/2}.$$

If we instead try to write it as a graph $x = G(y)$ we get a single graph

$$x = y^{2/3}.$$

This gives the Cartesian equation of the curve, or we can cube this relation to get rid of the cube root. Writing in standard form we have

$$x - y^{2/3} = 0, \qquad \text{or} \qquad x^3 - y^2 = 0.$$

Example. Consider the cycloid formed by a point on the circumference of a circle, as the circle rolls along a line without slipping. We derived the parametric equation for the cycloid

$$\begin{cases} x = r(\theta - \sin\theta), \\ y = r(1 - \cos\theta) \end{cases} \tag{10.1.1}$$

where θ is the angle through which the circle has turned, and we are interested in the behavior of the cycloid at it's lowest point. Does it have a loop? Is it's slope finite or infinite at this point? The lowest point of the cycloid corresponds to the value $\theta = 0$.

Tangents. Suppose that a curve is locally a graph over the x axis and given as a parametric curve with parameter t. Then y is a function of x which is a function of t. The chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Dividing both sides by dy/dt gives the equation for dy/dx,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \text{if} \quad \frac{dx}{dt} \neq 0 \tag{10.2.2}$$

If $dx/dt = 0$ but $dy/dt \neq 0$ then $dy/dx = \pm\infty$ so the tangent line is vertical.
Example. For the cycloid we have

$$\begin{cases} dx/d\theta = r(1 - \cos\theta) \\ dy/d\theta = r\sin\theta \end{cases},$$

so

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}.$$

Unfortunately at $\theta = 0$ we get $0/0$, so we can draw no conclusion. To study what happens as $\theta \downarrow 0$, we use l'hôpital's rule.

$$\lim_{\theta\downarrow 0}\frac{dy}{dx} = \lim_{\theta\downarrow 0}\frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta\downarrow 0}\frac{d(\sin\theta)/d\theta}{d(1 - \cos\theta)/d\theta} = \lim_{\theta\downarrow 0}\frac{\cos\theta}{\sin\theta} = +\infty$$