

Lecture 16. Tangent planes and linear approximations.

Suppose $f(x, y)$ is a function and consider the graph $z = f(x, y)$. We will derive the equation of the tangent plane to this graph at the point (x_0, y_0, z_0) , where $z_0 = f(x_0, y_0)$. Let C_1 and C_2 be the curves obtained by intersecting the surface with the planes $y = y_0$ and $x = x_0$ respectively. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at (x_0, y_0, z_0) . We learned last time that T_1 and T_2 have slopes $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ so the equations are

$$\begin{aligned} T_1 : \quad & (z - z_0) = f_x(x_0, y_0) (x - x_0), \\ T_2 : \quad & (z - z_0) = f_y(x_0, y_0) (y - y_0). \end{aligned}$$

The **tangent plane** to the graph at (x_0, y_0, z_0) contains the lines T_1 and T_2 . Since it is a plane, it has the equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ and after dividing by C and moving two terms to the other side we get

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

Setting $y = y_0$ we get the tangent line T_1 , so $a = f_x(x_0, y_0)$. Setting $x = x_0$ we get the tangent line T_2 so $b = f_y(x_0, y_0)$ and the equation of the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example. Find the tangent plane to $z = f(x, y) = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution. $f_x = 2x$, $f_y = 2y$. So $f_x(1, 2) = 2$, $f_y(1, 2) = 4$. So the tangent plane has equation $z - 5 = 2(x - 1) + 4(y - 2)$.

Example. Find the tangent plane to the surface $ze^z = x^2 - y^2$ at $(1, 1, 0)$.

Solution. We cannot write z explicitly in terms of x and y , but we can differentiate implicitly. First note that our independent variables are x and y and it is z which depends on these.

$$z_x e^z + z e^z z_x = 2x.$$

Plugging in $(x, y, z) = (1, 1, 0)$ this gives

$$z_x(1, 1) = 2.$$

Similarly,

$$z_y e^z + z e^z z_y = -2y,$$

and plugging in $(x, y, z) = (1, 1, 0)$ gives

$$z_y(1, 1) = -2.$$

The equation of the tangent plane is thus

$$z - 0 = 2(x - 1) - 2(y - 1).$$

Example. For the function $z(x, y)$ defined implicitly by $zx = x^2 + y^2$, use implicit differentiation to find the z_x and z_y at $(x, y) = (1, 1)$, and check by solving for z and differentiating directly.

Solution. Using the product rule, $(zx)_x = 2x$, so $z_x x + z = 2x$. Plugging in $(x, y) = (1, 1)$ to the equation $zx = x^2 + y^2$ we have $z(1, 1) = 1^2 + 1^2 = 2$ and so we get $z_x(1, 1) + 2 = 2$ so $z_x(1, 1) = 0$.

Similarly $(zx)_y = 2y$ and so $z_y = 2y$ and $z_y(1, 1) = 2$.

We can check these by writing $z = x + y^2/x$. Then $z_x = 1 - y^2/x^2$ and so $z_x(1, 1) = 0$, while $z_y = 2y/x$ so $z_y(1, 1) = 2$.

Linear Approximation. We can approximate f close to the fixed point (x_0, y_0) using the tangent plane, and we will give some notation. If Δx and Δy represent the change in x and y respectively, then $\Delta f(\Delta x, \Delta y)$ represents the change in the value of f , that is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

The quantities Δx and Δy are independent variables while Δf depends these. Now to make the notation consistent, we use dx, dy to denote the change in x and y respectively, then $df(dx, dy)$ denotes the change of the value of z on the tangent plane to the graph of f at the point (x_0, y_0) . That is

$$df = \frac{\partial f(x_0, y_0)}{\partial x} dx + \frac{\partial f(x_0, y_0)}{\partial y} dy.$$

When Δx and Δy are small, we have $\Delta f \approx df$. We can emphasize that we are comparing the in the value of $z = f(x, y)$ to the value of z on the tangent plane by writing $\Delta z = \Delta f$ and $dz = df$. Then the statement that the graph is close to the tangent plane is then $\Delta z \approx dz$.

Example. Estimate the value of the function given by $ze^z = x^2 - y^2$ at the point $(x, y) = (1.1, 1.2)$.

Solution. We have that $\Delta x = dx = 0.1$ and $\Delta y = dy = 0.2$. Using the values $z_x(1, 1) = 2$ and $z_y(1, 1) = -2$ which we computed earlier, we have

$$dz = z_x dx + z_y dy = 0.2 - 0.4 = -0.2.$$

Then

$$z = z_0 + \Delta z \approx z_0 + dz = 0 - 0.2 = -0.2.$$