Lecture 19: Maximum and minimum values.

Definition. A function $f(x,y)$ has a **local maximum** at (a,b) if $f(a,b) \geq f(x,y)$ when (x,y) is near (a,b). The number $f(a,b)$ is then called the **local maximum value**. The function f has a **global (absolute) maximum** at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in the domain of f. Then the value $f(a,b)$ is called the **global maximum** of f.

Similarly, $f(x,y)$ has a **local minimum** at (a,b) if $f(a,b) \leq f(x,y)$ when (x,y) is near (a,b), and then $f(a,b)$ is called the **local minimum value**. The function f has a **global minimum** at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) in the domain of f. Then $f(a,b)$ is the **global minimum**.

By an **extreme value** we mean a maximum or minimum.

Theorem. If $f(x,y)$ has as a local maximum or minimum at (a,b) then $f_x(a,b) = f_y(a,b) = 0$.
Proof. If f has a local max at (a,b) then the function of one variable $x \to f(x,b)$ has a local maximum at $x = a$, so $f_x(a,b) = 0$. Similarly the function of one variable $y \to f(a,y)$ has a local maximum at $y = b$, so $f_y(a,b) = 0$.

Remark. The conditions $f_x(a,b) = f_y(a,b) = 0$ are equivalent to the condition that $\nabla f(a,b) = 0$. Geometrically the condition means that the tangent plane to the graph $z = f(x,y)$ is horizontal at (a,b).

Definition. A point (a,b) is called a **critical point** of $f(x,y)$ if $f_x(a,b) = f_y(a,b) = 0$.
By the above theorem a local maximum or local minimum has to be a critical point. However, not all critical points are local maxima or minima. A critical point which is not a local maxima or minima is called a **saddle point**. For functions of one variable take for example $f(x) = x^3$ at $x = 0$.

How do we know if a critical point is a local maximum, a local miminum or a saddle point? For functions of one variable, if $f'(a) = 0$ and $f''(a) > 0$ then it is a min and if $f''(a) < 0$ then it is a max and if $f''(a) = 0$ then it could be a max, min or saddle.

Theorem (The second derivative test). Suppose that (a,b) is a critical point and

$$D = f_{xx}(a,y)f_{yy}(a,b) - f_{xy}(a,b)^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

If $D > 0$ and $f_{xx}(a,y) > 0$ then $f(a,b)$ is a local minimum.
If $D > 0$ and $f_{xx}(a,y) < 0$ then $f(a,b)$ is a local maximum.
If $D < 0$ then $f(a,b)$ is called a saddle point - it is neither a local maximum nor a local minimum.
If $D = 0$ then the test is inconclusive.

Classic Examples. Using the test, find and classify the critical points of

(a). $f(x,y) = x^2 + y^2$.
(b). $f(x,y) = x^2 - y^2$.
(c). $f(x,y) = -x^2 - y^2$.
(d). $f(x,y) = 0$.

Solution.

(a). Critical point $(0,0)$, $f_{xx}(0,0) = 2$, $D = 4$. Local min. (In fact we know it's a global min.)

(b). Critical point $(0,0)$, and here $f_{xx}(0,0) = 2$, $D = -4$. Saddle.

(c). Critical point $(0,0)$, $f_{xx}(0,0) = -2$, $D = 4$. Local max. (In fact we know it's a global max.)

(d). Every point is critical, local max and min and global max and min.

Theorem. (Extreme value theorem) If f is continuous on a closed and bounded set D then f attains an absolute maximum and an absolute minimum in D.

To find the max and min in a closed and bounded domain:

1) Find the critical points in the domain.
2) Find the extreme values on the boundary.
3) Find the max and min of the function at these points.

We do not need to use the second derivative test since by the extreme value theorem we know that there is a max and a min and any max or min either has to be a point on the boundary or a critical point in the interior.