Math 21C Calculus and Analytic geometry

Lecture 2: Tangents, area under a curve, and arclength.
Example. Sketch the curve

$$\begin{cases} x = t(t^2-3) \\ y = 3(t^2-3) \end{cases}.$$

We have

$$\begin{cases} dx/dt = 3(t^2-1) \\ dy/dt = 6t \end{cases}.$$

The points where $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$ are of interest because at these points the tangent is horizontal or vertical and as you pass through these points the tangent can switch to point in a different quadrant, so we calculate $dx/dt = 0$ at $t = \pm 1$ and $dy/dt = 0$ at $t = 0$.

t	$t < -1$	-1	$-1 < t < 0$	0	$0 < t < 1$	1	$1 < t$
dx/dt	$+$	0	$-$	$-$	$-$	0	$+$
dy/dt	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$\rightarrow$	$\cdot$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\cdot$	$\rightarrow$
y	$\downarrow$	$\downarrow$	$\downarrow$	$\cdot$	$\uparrow$	$\uparrow$	$\uparrow$
curve	$\searrow$	$\downarrow$	$\swarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$

Here for example, for $t > 1$, "$\nearrow$" means that as t increases in the , the tangent to the curve points somewhere between due north and due east. (It may change as t changes, but it must stay within this range.) Computing the values

t	-1	0	1
(x, y)	$(2, -6)$	$(0, -9)$	$(-2, -6)$

enables us to make a sketch of the curve, but to really understand what happens for $t < -1$ and $t > 1$, we need to take a closer look at the parametric equations.

The sign of d^2y/dx^2 enables us to determine when the graph is concave up or down. In general for a parametric curve we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{dx/dt}\frac{d}{dt}\left(\frac{dy}{dx}\right).$$

Applying this in our case we have

$$\frac{d^2y}{dx^2} = \frac{1}{3(t^2-1)}\frac{d}{dt}\left(\frac{2t}{t^2-1}\right) = \frac{1}{3(t^2-1)}\left(\frac{2}{t^2-1} - \frac{2t\,2t}{(t^2-1)^2}\right) = -2\frac{t^2+1}{(t^2-1)^3}.$$

We can add the final row to our table:

t	$t<-1$	-1	$-1<t<0$	0	$0<t<1$	1	$1<t$
curve	$\frown$	$\cdot$	$\smile$	$\smile$	$\smile$	$\cdot$	$\frown$

This enables us to sketch the curve pretty well. In particular we see that there is a point where the curve crosses itself, indeed there is some $t_- < -1$ and some $t_+ > 1$ which give the same point (x, y). To determine the value of t for which this happens, we get the equations

$$\begin{aligned} t_-(t_-^2 - 3) &= t_+(t_+^2 - 3), \\ 3(t_-^2 - 3) &= 3(t_+^2 - 3). \end{aligned}$$

Solving the second equation gives $t_-^2 = t_+^2$ and so $t_- = -t_+$. Plugging this into the first equation gives $-t_+(t_+^2 - 3) = t_+(t_+^2 - 3)$ and so $t_+(t_+^2 - 3) = 0$. Since $t_+ > 1$ we get the solution $t_+ = \sqrt{3}$, so $t_- = -\sqrt{3}$ and $(x, y) = (0, 0)$.

Area under a curve. For a graph $y = F(x)$ with $x_0 \le x \le x_1$, the area under the graph and above the x-axis is given by

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} F(x)\,dx.$$

If x and y are given by a parametric equation in terms of a parameter t with $t_0 \le t \le t_1$, we can change variables to obtain that the area under the curve is

$$\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt.$$

For example, we computed the area under one arch of the cycloid given in 10.2.2 by following Example 3 of 10.2.

Arclength. In section 8 of the book there was a formula for the arc length L of a curve in the form of a graph (10.1.1): $L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$.
If we now also express the same curve in parametric form (10.1.3), use (10.2.1) and a change of variables, $dx = (dx/dt)dt$, in the integral we obtain:

(10.3.1)
$$L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

We computed the arclength of one arch of the cycloid given in 10.2.2 by following Example 2 of 10.3.