

Lecture 26: Triple Integrals.

Geometric Interpretation. Our old friend the single-variable integral can be used to compute arc-length of curves and area of regions in the plane. Double integrals can be used to compute the surface area of surfaces and the volume of solid regions in space. How about triple integrals? Well they can be used to compute the volume of curved 3-dimensional spaces and the 4-dimensional volume of 4-dimensional spaces. Although these are objects of great importance to geometers and physicists, they are not easy to visualize or understand. There are two physical applications of triple integrals which are easy to understand.

Application 1. A solid region E is made of a material which has density $f(x, y, z)$ at the point (x, y, z) . (measured in units of mass per unit of volume). Then $\iiint_E f(x, y, z) dV$ represents the total mass of E . There are similar applications of the double and single integrals which we won't go into.

Application 2. The integral $\iiint_E q dV$ computes the volume of the solid region S .

Example. Calculate $\iiint_T y dV$ where T is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$.

Solution. Is on the final solutions f3.

Example. Let E be the solid region bounded by the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $z = x^2 + y^2$. Calculate

- (a). $\iiint_E z dV$,
- (b). The volume of E .
- (c). The surface area of E .

Solutions. We first need to describe the region E . We make a sketch and see that it is a region of Type 1. We project it onto the xy plane to get a disc D centered at the origin $(0, 0)$. The radius of this disc is the same as the radius of the circle of intersection of the sphere and the paraboloid which is given by $2 = x^2 + y^2 + z^2 = 2z^2$, so $z = 1$ and $x^2 + y^2 = 1$ the radius is 1. Then E is described by

$$\begin{cases} (x, y) \in D \\ x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2} \end{cases}$$

We describe the region D using polar coordinates.

(a).

$$\begin{aligned} \iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z dz dA &= \iint_D \frac{z^2}{2} \Big|_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dA = \int_D \frac{2 - x^2 - y^2 - (x^2 + y^2)^2}{2} dA \\ &= \int_0^{2\pi} \int_0^1 \frac{2 - r^2 - r^4}{2} r dr d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{8} - \frac{r^6}{12} \right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left(1 - \frac{1}{8} - \frac{1}{12} \right) d\theta = \frac{19\pi}{12}. \end{aligned}$$

(b). We can set this up as a triple or double integral. As a double integral we get

$$\begin{aligned} \iint_D 2-x^2-y^2-(x^2+y^2)^2 dA &= \int_0^{2\pi} \int_0^1 2-r^2-r^4 drd\theta = \int_0^{2\pi} \left(r^2 - \frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left(1 - \frac{1}{4} - \frac{1}{6} \right) d\theta = \frac{7\pi}{6}. \end{aligned}$$

As a triple integral we have

$$\iiint_E 1 dV = \iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} 1 dz dA = \iint_D 2-x^2-y^2-(x^2+y^2)^2 dA,$$

which we see is the double integral we just calculated.

(c). There are two pieces of surface, the sphere and the parabola and we must sum them up. For the sphere we have $z = \sqrt{2-x^2-y^2}$ and

$$\begin{aligned} \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dA \\ &= \iint_D \sqrt{1 + \left(-\frac{x}{\sqrt{2-x^2-y^2}} \right)^2 + \left(-\frac{y}{\sqrt{2-x^2-y^2}} \right)^2} dA \\ &= \int_0^{2\pi} \int_0^1 \sqrt{1 + \frac{r^2}{2-r^2}} r drd\theta = \int_0^{2\pi} \frac{r\sqrt{2}}{\sqrt{2-r^2}} drd\theta \\ &= -\frac{2\pi\sqrt{2}}{3} (2-r^2)^{3/2} \Big|_0^1 = \frac{2\pi\sqrt{2}(2^{3/2}-1)}{3} \end{aligned}$$

For the parabola we have $z = x^2 + y^2$ and

$$\begin{aligned} \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dA \\ &= \iint_D \sqrt{1 + 4x^2 + 4y^2} dA = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} r drd\theta \\ &= \frac{2\pi}{3} (1+r^2)^{3/2} \Big|_0^1 = \frac{2\pi(2^{3/2}-1)}{3} \end{aligned}$$

Summing these, we get that the surface area of E is

$$\frac{2\pi(2^{1/2}+1)(2^{3/2}-1)}{3}.$$