Lecture 3: Arclength and surface area. The length of a parametric curve with parameter t, $a \le t \le b$ is

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt. \tag{10.3.2}$$

We will now show why this is true. Suppose the parametric equation is

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}.$$

We take points $t_0, t_1, \dots, t_n$ in the interval $[a, b]$: $a = t_0 < t_1 < ... < t_{i-1} < t_i < ... < t_n = b$ and consider the points $P_i == (x_i, y_i) = (f(t_i), g(t_i))$ on the curve. Let l_i be the length of the line segment joining P_i to P_{i+1}, then

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \ell_i,$$

where the number of points, $n+1$, tend to infinity in such a way that the maximum distance $t_i - t_{i-1}$ between neighboring points tends to 0. (At each stage when we throw in more points, the indices i of t_i are reassigned so that the values t_i are always in increasing order!)

To see why L equals (10.3.2), consider the change in t, x and y as t goes from t_i to t_{i+1}:

$$\triangle t_i = t_{i+1} - t_i, \qquad \triangle x_i = x_{i+1} - x_i, \qquad \triangle y_i = y_{i+1} - y_i.$$

By the Pythagorean theorem

$$\ell_i = \sqrt{\triangle x_i^2 + \triangle y_i^2},$$

Since by the linear approximation

$$\triangle x_i \approx \frac{dx(t_i)}{dt} \triangle t_i, \qquad \triangle y_i \approx \frac{dy(t_i)}{dt} \triangle t_i,$$

so

$$\ell_i \approx \sqrt{\left(\frac{dx(t_i)}{dt}\right)^2 + \left(\frac{dy(t_i)}{dt}\right)^2}\, \triangle t_i$$

and in fact in the limit we get

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{dx(t_i)}{dt}\right)^2 + \left(\frac{dy(t_i)}{dt}\right)^2}\, \triangle t_i,$$

which equals the integral in 10.3.2 since we have a limit of Riemann sums which approximate the integral.

Example. We found the arc length of the circle of radius R, following Example 1 of 10.3. For example, if we parameterize the circle as

$$\begin{cases} x = R\cos 2\theta \\ y = R\sin 2\theta \end{cases}, \qquad 0 \le \theta \le 2\pi,$$

we get that the length is

$$L \;=\; \int_0^{2\pi} \sqrt{(-2R\sin 2\theta)^2 + (R\cos 2\theta)^2}\, d\theta \;=\; \int_0^{2\pi} 2R\, d\theta \;=\; 4\pi R.$$

The reason that this is double the actual length of the circle is that the parameterization we gave wound twice around the circle. If we want to find the length of a curve we should in general parameterize it in such a way that each point is passed through exactly once. (More generally, the total length of the points which are not passed through exactly once should be zero.) In general the length of a curve is independent of the choice of parametrization as long as we observe this rule.

Polar coordinates. The polar coordinates (r, θ) of the point (x, y) in rectangular coordinates (x, y) are defined by

$$\begin{cases} x = r\,\cos\theta, \\ y = r\,\sin\theta \end{cases}$$

so

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}.$$

Note that (r, θ) corresponds to the same point in the $x - y$ plane as $(r, \theta + 2n\pi)$ and $(-r, (2n + 1)\pi)$, where n is an integer. A curve in polar coordinates:

$$x = r\,\cos\theta, \qquad y = r\,\sin\theta, \qquad r = f(\theta).$$

is just a special case of a parametrized curve.

Example. Sketch the curve $r = 2\sin\theta$ for $0 \le \theta \le \pi$, calculate it's slope at $\theta = \pi/4$, and compute it's length.
By first sketching r as a graph of θ in the $\theta - r$ plane. (In fact the curve is a circle of radius 1 centered at $(0, 1)$. In fact, multiplying by r gives $r^2 = 2r\,\sin\theta$ which is the same as $x^2 + y^2 = 2y$ which if we complete the square is equivalent to $x^2 + (y - 1)^2 = 1$.)
Tangents to polar curves To calculate the slope of the tangent we use the formula (10.2.1):

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

Remember here that r is a function of θ on the curve, and use the product rule to differentiate.
Applying this when $r = 2\sin\theta$ gives

$$\frac{dy}{dx} = \frac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos\theta\cos\theta - 2\sin\theta\sin\theta} = \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}.$$

Plugging in the point $\theta = \pi/4$ gives

$$\frac{dy}{dx} = \frac{1}{1/\sqrt{2} - 1/\sqrt{2}} = \frac{1}{0} = \pm\infty.$$

One can check on the sketch that the tangent is indeed vertical at $\pi/4$.
Arc length in polar coordinates. By plugging in the polar curve in the formula for the arc length (10.3.2) and calculating

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 + \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 = ... = \left(\frac{dr}{d\theta}\right)^2 + r^2,$$

we get

$$L = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \tag{10.4.3}$$

Example. Find the arc length of the polar curve $r = 2\sin\theta$, where $0 \le \theta \le \pi$.
We have $L = \int_0^\pi \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta = 2\int_0^\pi d\theta = 2\pi$.

Area of a surface of revolution. If a parametric curve with parameter t, $a \le t \le b$ is revolved around the x-axis, the area of the surface obtained is

$$A = \int_a^b 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$