Lecture 5: Vectors. Imagine driving a car along a winding road. At each point the velocity $\mathbf{v}$ is the speed and direction in which you are driving. It can be represented by an arrow pointing in the direction in which you are driving and whose length is the speed. Such a quantity which has both magnitude and direction is called a **vector**.

Definition. A two dimensional **vector a** is an ordered pair of real numbers $\langle a_1, a_2\rangle$ called components. A three dimensional **vector a** is an ordered triple of real numbers $\langle a_1, a_2, a_3\rangle$ called components. A representative of the vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is a directed line segment from a point $A(x, y, z)$ to a point $B(x+a_1, y+a_2, z+a_3)$.
Example. What is the vector which is represented by the directed line segment from $(4, 3, 2)$ to $(1, 4, 4)$?
Solution. $\langle 1-4, 4-3, 4-2\rangle = \langle -3, 1, 2\rangle$.

The **position vector** of the point $P(a_1, a_2, a_3)$ is the vector $\langle a_1, a_2, a_3\rangle$. It is represented by the directed line segment from the origin O to P.
Length. The length or magnitude of a 2-dimensional vector $\mathbf{a} = \langle a_1, a_2\rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. The length of a 3-dim. vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Adding vectors. The sum of the two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

Multiplying vectors by scalars. $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, c is a number (scalar) then

$$c\mathbf{a} \;=\; \langle ca_1, ca_2, ca_3\rangle.$$

The vectors **a** and **b** are **parallel** if either $\mathbf{a} = \mathbf{0}$, (the zero vector is parallel to every other vector) or $\mathbf{b} = c\mathbf{a}$ for some constant c.

Example. $\mathbf{a} = \langle -7, 2, 4\rangle$, $\mathbf{b} = \langle 1, 3, -7\rangle$. Compute $\mathbf{a}+\mathbf{b}$, $2\mathbf{b}$, $\mathbf{a} - 2\mathbf{b}$ and $|\mathbf{a}|$.
Solution. $\mathbf{a} + \mathbf{b} = \langle -6, 5, -3\rangle$, $2\mathbf{b} = \langle 2, 6, -14\rangle$, $\mathbf{a} - 2\mathbf{b} = \langle -9, -4, 18\rangle$, $|\mathbf{a}| = \sqrt{(-7)^2 + 2^2 + 4^2} = \sqrt{49 + 4 + 16} = \sqrt{69}$.

Geometric Interpretation. These operations have a geometric interpretation. Triangle law for addition: Represent the vectors **a** and **b** by arrows so that the initial point of **b** equals the terminal point of **a**. Then $\mathbf{a} + \mathbf{b}$ is represented by the arrow from the initial point of **a** to the terminal point of **b**.

In particular we see that $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ is in the same direction as **a** and twice as long.

In general if $c > 0$, the vector $c\mathbf{a}$ points in the same direction as **a** but is c times as long.

If $c < 0$, the vector $c\mathbf{a}$ points in the opposite direction to **a** and is $|c|$ times as long. In particular, $-\mathbf{a}$ is the same length as **a** but points in the opposite direction.

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ also has a geometric interpretation. If $\mathbf{a}$ and $\mathbf{b}$ are represented by arrows with the same initial point, then $\mathbf{a} - \mathbf{b}$ points from the terminal point of $\mathbf{b}$ to the terminal point of $\mathbf{a}$.

You should compare this with the fact that the arrow with initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2) represents the vector $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

The standard **basis** vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$.
Any vector can be written in terms of these: For example, $\langle 5, -1, 2 \rangle = 5\langle 1, 0, 0 \rangle - \langle 0, 1, 0 \rangle + 2\langle 0, 0, 1 \rangle = 5\mathbf{i} - \mathbf{j} + 2\mathbf{j}$. In general $\langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{j}$.

A **unit vector u** is a vector of length one, that is $|\mathbf{u}| = 1$. The unit vector in the direction $\mathbf{a}$ is $\mathbf{a}/|\mathbf{a}|$. For example the unit vector in the direction $\langle -7, 2, 4 \rangle$ is $\langle -7, 2, 4 \rangle/\sqrt{69} = \langle -7/\sqrt{69}, 2/\sqrt{69}, 4/\sqrt{69} \rangle$.

The dot product (scalar product) (inner product). $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ the dot product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example. $\mathbf{a} = \langle -7, 1, 3 \rangle$ and $\mathbf{b} = \langle 2, -7, 3 \rangle$ then

$$\mathbf{a} \cdot \mathbf{b} \;=\; -7(2) \;+\; 1(-7) \;+\; 3(3) \;=\; -14 - 7 + 9 \;=\; -12.$$

Geometric Interpretation. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\mathbf{a} \cdot \mathbf{b} \;=\; |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta.$$

Example. Calculate the angle between $\mathbf{a}$ and $\mathbf{b}$ from the previous example.
Solution.

$$\cos\theta \;=\; \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \;=\; \frac{-12}{\sqrt{49 + 1 + 9}\sqrt{4 + 49 + 9}} = \frac{-12}{\sqrt{59}\sqrt{62}} = -0.198...$$

so $\theta = \cos^{-1}(-0.198...) = 101.14°$.

To derive the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we need the cosine formula which states

$$|\mathbf{a} - \mathbf{b}|^2 \;=\; |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta.$$

We derived this cosine rule by pythagoras rule.

$$|\mathbf{a} - \mathbf{b}|^2 \;=\; (|\mathbf{a}| - |\mathbf{b}|\cos\theta)^2 \;+\; |\mathbf{b}|^2\sin^2\theta \;=\; |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta.$$