MATH 21C Solutions to Midterm 1 Winter 2001, Tomter.

1. $\sin t = 0$ and $1 + \sin t \cos t = 1$ when $t = 0$ and $t = \pi$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos^2 t - \sin^2 t}{\cos{(t)}}$$

When $t = 0$ then $dy/dx = 1$ and when $t = \pi$ then $dy/dx = -1$.
(a) The equations for the tangents are therefore $y - 1 = x$ repectively $y - 1 = -x$.
(b) $\mathbf{a} = \langle 1, 1 \rangle$ and $\mathbf{b} = \langle 1, -1 \rangle$ are parallel to the tangents. Since $\mathbf{a} \cdot \mathbf{b} = 0$ it follows that the two lines are perpendicualr, i.e the angle is $\pi/2$.

2. Rewriting the curve we get $x = \frac{1}{2}\cos{(2t)}$, $y = 1 + \frac{1}{2}\sin{(2t)}$, $t \in [0, \frac{\pi}{2}]$.
(a) The tangent is horizontal when $\dfrac{dy}{dt} = \cos 2t = 0$, i.e. $t = \frac{\pi}{4}$ and $(x, y) = (0, \frac{3}{2})$.
(b) The area of the surface of revolution around the x-axis is

$$A = \int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^{\pi/2} 2\pi(1 + \frac{1}{2}\sin 2t)\sqrt{\cos^2 2t + \sin^2 2t}\, dt$$
$$= \int_0^{\pi/2} \pi(2 + \sin 2t)\, dt = \pi\big(2t - \frac{1}{2}\cos 2t\big)\Big|_0^{\pi/2} = \pi\big(\pi + 1\big)$$

3. (a) $\mathbf{a} = \overrightarrow{PQ} = \langle -2, 1, 0 \rangle$ and $\mathbf{b} = \overrightarrow{PR} = \langle -2, 0, -5 \rangle$ are parallel to the plane so

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 0 \\ -2 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ -2 & 0 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 10\mathbf{j} + 2\mathbf{k},$$

is normal.
(b) $P(2, 0, 0)$ is in the plane so the eq. of the plane is $-5(x - (-2)) - 10(y - 0) + 2(z - 0) = 0$ or $-5x - 10y + 2z + 10 = 0$. To find the point on the line that intersects the plane we plug the parametric eqautions of the line into the eqaution for the plane: $-5(1 + t) - 10(1 - t) + 2(-3 - 3t) + 10 = 0$ or $t = -11$ which gives $(x, y, z) = (-10, 12, 30)$.

4. (a) That $P(2, 0, -3)$, $Q(0, 5, 4)$, $R(1, 1, -1)$ and $S(5, -12, -18)$ lie in a plane is equivalent to that $\mathbf{a} = \overrightarrow{PQ} = \langle -2, 5, 7 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1, 1, 2 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 3, -12, -15 \rangle$ are parallel to a plane. This is the case since

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -2 & 5 & 7 \\ -1 & 1 & 2 \\ 3 & -12 & -15 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ -12 & -15 \end{vmatrix} - 5\begin{vmatrix} -1 & 2 \\ 3 & -15 \end{vmatrix} + 7\begin{vmatrix} -1 & 1 \\ 3 & -12 \end{vmatrix} = 0$$

(b) Following the hint we want to find the distance betwen P and the plane $x - y + z - 2 = 0$ which is $D = |2 - 0 - 3 - 2|/\sqrt{1^2 + 1^2 + 1^2} = 3/\sqrt{3} = \sqrt{3}$.