

**Solutions for Math 21C Midterm II, Fall 02. Lindblad.**

1. (a) Velocity  $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + 3t^{1/2}/2 \mathbf{k}$  and speed

$$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + (3/2)^2 t} = \sqrt{1 + (3/2)^2 t}.$$

Hence  $\mathbf{r}'(\pi) = -\mathbf{j} + (3/2)\sqrt{\pi} \mathbf{k}$  and  $|\mathbf{r}'(\pi)| = \sqrt{1 + (3/2)^2 \pi}$ .

(b) The distance traveled is the arc length:

$$\int_0^{10} |\mathbf{r}'(t)| dt = \int_0^{10} \sqrt{1 + (3/2)^2 t} dt = (1 + (\frac{3}{2})^2 t)^{3/2} (\frac{2}{3})^3 \Big|_0^{10} = ((1 + (\frac{3}{2})^2 10)^{3/2} - 1) (\frac{2}{3})^3.$$

2. (a)  $\nabla F(x, y) = \langle 3x^2 - 2x, 2y - 1 \rangle$ .

(b) Unit vector in the direction:  $\mathbf{u} = \langle 1, 2 \rangle / |\langle 1, 2 \rangle| = \langle 1, 2 \rangle / \sqrt{5}$ .  $\nabla F(2, 3) = \langle 8, 5 \rangle$ .

The directional derivative  $D_{\mathbf{u}}F(2, 3) = \nabla F(2, 3) \cdot \mathbf{u} = \langle 8, 5 \rangle \cdot \langle 1, 2 \rangle / \sqrt{5} = 18/\sqrt{5}$ .

(c) By the chain rule  $h'(t) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$  so  $h'(0) = \nabla F(1, 0) \cdot \langle 1, 1 \rangle = \langle 1, -1 \rangle \cdot \langle 1, 1 \rangle = 0$

3. (a)  $\langle f_x, f_y \rangle = \langle x, 3y \rangle / \sqrt{x^2 + 3y^2}$  so  $\langle f_x(1, 1), f_y(1, 1) \rangle = \langle 1, 3 \rangle / 2$  and the equation of the tangent plane is  $z - 2 = (x - 1)/2 + 3(y - 1)/2$ .

(b)  $z - 2 = 0.1/2 + 3 \cdot 0.2/2 = 0.35$  and the linear approximation is  $z = 2 + 0.35 = 2.35$ .

4.  $f_x = 3x^2 - 2x = 0$  and  $f_y = 2y - 1 = 0$  gives  $(x, y) = (0, 1/2)$  or  $(x, y) = (2/3, 1/2)$ .

$f_{xx} = 6x - 2$ ,  $f_{yy} = 2$  and  $f_{xy} = 0$  and  $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4(3x - 1)$ .

If  $(x, y) = (0, 1/2)$  then  $D = -4 < 0$  so it is a saddle point.

If  $(x, y) = (2/3, 1/2)$  then  $D = 4 > 0$  and  $f_{xx} = 2 > 0$  so it is a local min.

5. Let  $F(x, y, z) = d(x, y, z)^2 = x^2 + y^2 + (z + 2)^2$  be the square of the distance from  $(0, 0, -2)$  to  $(x, y, z)$  and let  $G(x, y, z) = z^2 - x^2 - y^2 - 1$ .

We want to maximize  $F(x, y, z)$  subject to the constraint  $G(x, y, z) = 0$  and  $z > 0$ .

$\nabla F(x, y, z) = \langle 2x, 2y, 2(z + 2) \rangle$  and  $\nabla G(x, y, z) = \langle -2x, -2y, 2z \rangle$  so we must find all solutions to

$$2x = -\lambda 2x, \quad 2y = -\lambda 2y, \quad 2(z + 2) = \lambda 2z, \quad \text{and} \quad z^2 - x^2 - y^2 - 1 = 0$$

We see that if  $\lambda \neq -1$  then the first two equations gives  $x = y = 0$  and the third equation can not be solved if  $\lambda = 1$ : (i) If  $\lambda = 1$  no solutions. (ii)  $\lambda = -1$  then the first and second equation hold and the third equation gives  $z = -1$  so we get  $(x, y, -1)$  and  $G(x, y, -1) = -x^2 - y^2 = 0$  gives  $x = y = 0$  so the point is  $(0, 0, -1)$  and  $F(0, 0, -1) = 1$ . If  $\lambda \neq -1$  and  $\lambda \neq 1$  then  $(x, y, z) = (0, 0, 1/(1 + \lambda))$  and  $G(0, 0, 1/(1 + \lambda)) = 1/(1 + \lambda)^2 - 1 = 0$  gives  $\lambda = 0$  so the point is  $(0, 0, 1)$  and  $F(0, 0, 1) = 9$ . In conclusion the closest point is  $(0, 0, 1)$ .