MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 1: The connection between elliptic second order linear partial differential equations and Riemannian geometry.

This lecture: Elliptic second order partial differential operators. The principal symbol. The cotangent space and the tangent space. The principal symbol gives a metric, in other words a coordinate independent way of measuring the lengths of curves.

Let $U \subset \mathbb{R}^n$ be an open set. For $f : U \to \mathbb{R}$, we consider the operator L given by

$$Lu \;=\; -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} \;+\; \sum_{i=1}^{n} b^i(x) u_{x_i} \;+\; c(x)u. \tag{1}$$

We can assume $a^{ij} = a^{ji}$. We will generally make some smoothness assumptions on the coefficients a^{ij}, b^i, c. For now assume that **the coefficients are smooth** (C^∞). Clearly L is second order and linear ($L(u + tv) = L(u) + tL(v)$).

Definition. We say that L is **elliptic** if for each $p \in U$ there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(p)\xi_i\xi_j \;\geq\; \theta|\xi|^2, \qquad \text{for every } \xi \in \mathbb{R}^n. \tag{2}$$

If $\theta > 0$ can be chosen independently of $p \in U$ then we say that L is **uniformly elliptic** on U. The left hand side is called the *principal symbol* of the operator L.

Remark. The ellipticity condition is saying that the bilinear form defined by $a^{ij}(p)$ is positive definite. By the continuity of the a^{ij}, the value $\theta > 0$ can always be chosen to be uniform on compact subsets of Ω.

Example. The Laplacian (or negative the Laplacian depending on your convention)

$$\Delta u \;=\; -\sum_{i=1}^{n} u_{x_i x_i}$$

is (uniformly) elliptic on any open set. Some theory of the Laplacian can be found in Section 2.2 of the book. We are going to generalize some aspects of that theory to general elliptic operators.

Change of variables. How does the operator L change if we change variables? We have

$$\frac{\partial}{\partial x_i} \;=\; \sum_k \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}.$$

It is most convenient to use the notation of physicists and geometers. If Ω is a subset of Euclidean space and $y = (y^1, \ldots, y^n) : \Omega \to V \subset \mathbb{R}^n$ is a diffeomorphism, then for a function $u : \Omega \to \mathbb{R}$, we define

$$u_{y_k} \;:=\; \frac{\partial u}{\partial y_k} \;=\; (D_k(u \circ y^{-1})) \circ y.$$

We make this definition so that we can change coordinates without changing the name of the function. Indeed, if $x : \Omega \to U$ and $y : \Omega \to V$ are two coordinate maps on Ω, then

$$\frac{\partial}{\partial x_i} = \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}. \tag{3}$$

Suppose now that $L : C^\infty(\Omega) \to C^\infty(\Omega)$ is an operator which is given by (1), then we say that a^{ij} is the principal symbol of L in the x-coordinates and L is elliptic in the x-coordinates if (2) holds for each $p \in \Omega$. Changing to the y-coordinates, we have

$$Lu = \sum_{k,\ell}\left(\sum_{i,j}\frac{\partial y_k}{\partial x_i}a^{ij}\frac{\partial y_\ell}{\partial x_j}\right)u_{y_k y_\ell} + \text{ lower order terms.}$$

We notice that although the principal symbol of L changes when you change variables, it has a simple transformation law. Writing b^{ij} for the principal symbol of L in the y-coordinates,

$$b^{k\ell} = \sum_{i,j}\frac{\partial y_k}{\partial x_i}a^{ij}\frac{\partial y_\ell}{\partial x_j}. \tag{4}$$

Lemma. *If L is elliptic in the x-coordinates then it is elliptic in the y-coordinates.*

Assume L is elliptic in the x-coordinates and satisfying (2) and compute

$$\sum_{k,\ell} b^{k\ell}\eta_k\eta_\ell = \sum_{i,j}a^{ij}\left(\sum_k\frac{\partial y_k}{\partial x_i}\eta_k\right)\left(\sum_\ell\frac{\partial y_\ell}{\partial x_j}\eta_\ell\right) = \sum_{i,j}a^{ij}\xi_i\xi_j \geq \theta|\xi|^2,$$

where

$$\xi_i = \sum_k \frac{\partial y_k}{\partial x_i}\eta_k.$$

Since x and y are diffeomorphisms, the matrix dy/dx is invertible, and at each point in Ω we can find $\varepsilon > 0$ such that

$$|\xi| \geq \varepsilon|\eta|.$$

Thus if L is elliptic in one coordinate system, it is elliptic in any other. The same is not true for uniform ellipticity, because dy/dx could shrink to zero at the boundary of Ω.

What exactly is the principal symbol of L?

Definition. If V is a finite dimensional vector space, then a map $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ is a **bilinear form on** V if it is linear in each variable, that is $\langle u + tv, w\rangle = \langle u, w\rangle + t\langle v, w\rangle$ and similarly in the other variable. It is **symmetric** if $\langle u, v\rangle = \langle v, u\rangle$

for all $u, v \in V$. A **quadratic form on** L is a map $q : V \to \mathbb{R}$ which is given by $q(v) = \langle v, v\rangle$ for some bilinear form.

If we fix coordinates x on Ω then the principal symbol of L it is a real quadratic form on $\mathbb{R}^n$. If we change coordinates then the quadratic form changes by the rule (4). There is a way to make a cleaner definition of the principal symbol which makes the transformation rule easier to remember. We notice using (4) and the chain rule that if u, v are $C^1(\Omega)$ then

$$\sum_{k,\ell} b^{k\ell} u_{y_k} v_{y_\ell} \;=\; \sum_{i,j} a^{ij} u_{x_i} v_{x_j}. \tag{5}$$

Definition. The **cotangent space to Ω at** p, also known as **the space of differentials** at p is an n dimensional vector space denoted by $T_p^*(\Omega)$. If $x = (x_1, \ldots, x_n)$ are coordinates on Ω then a basis for $T_p^*(\Omega)$ is given by $dx_1|_p, \ldots, dx_n|_p$, so the general cotangent vector has the form $a_1 dx_1|_p + \cdots + a_n dx_n|_p$. If $f \in C^1(\Omega)$ then $df|_p$ is a differential at p defined by

$$df|_p \;=\; \sum_{i=1}^{n} f_{x_i}(p) dx_i|_p.$$

This definition is consistent. We can think of differentials as linear approximations to functions which vanish at p. (Strictly speaking we can define a differential to be an equivalence class of functions $f \in C^1(\Omega)$ with $f(p) = 0$, where the equivalence relation is "$f \sim h$ if the linear approximations of f and h at p agree in every system of coordinates".)

We define a symmetric bilinear form on $T_p^*(\Omega)$ by

$$\langle df, dh\rangle \;=\; \sum_{i,j} a^{ij} f_{x_i} h_{x_j}.$$

Notice that

$$\langle dx_i, dx_j\rangle \;=\; a^{ij}.$$

Then because of (5), this definition is independent of the choice of coordinates. We can define the principal symbol of L at p to the quadratic form coming from this bilinear form, so

$$\sigma(L)\left(\sum_i \xi_i dx_i\right) \;=\; \sum_i a^{ij} \xi_i \xi_j.$$

Definition. The **tangent space to Ω at** p, also known as the space of derivations at p or the space of directional derivatives at p is an n dimensional vector space denoted by $T_p(\Omega)$. If $x = (x_1, \ldots, x_n)$ are coordinates on Ω then a basis for $T_p(\Omega)$ is

$$\left.\frac{\partial}{\partial x_1}\right|_p, \;\ldots,\; \left.\frac{\partial}{\partial x_n}\right|_p.$$

The general tangent vector has the form

$$a_1 \frac{\partial}{\partial x_1}\bigg|_p + \ldots + a_n \frac{\partial}{\partial x_n}\bigg|_p .$$

The inverse of the principal symbol of L at p defines a bilinear form on the tangent space to Ω at p. This is what is known as a Riemannian **metric** and it can be used to measure the length of curves on Ω. Length minimizing curves or geodesics are important in the study of L.