MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 10. Smooth Approximation.

Last time: The norm on the Sobolev space $W^{k,p}(U)$ for $1 \le p < \infty$ is

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \int_U |D^\alpha f(y)|^p \, dy \right)^{1/p}.$$

We will approximate functions in Sobolev spaces by smooth functions. When we want to prove a statement concerning Sobolev spaces we will often be able to prove it for smooth functions and then extend by continuity. This will save us having to deal constantly with weak derivatives.

Definitions. $W_0^{k,p}(U)$ is the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

$$W_{\text{loc}}^{k,p}(U) = \{u \in L_{\text{loc}}^1(U) : u|_V \in W^{k,p}(V) \text{ for all } V \subset\subset U\}.$$

This is a topological space with $u_j \to u$ in $W_{\text{loc}}^{k,p}(U)$ if and only if $u_j \to u$ in $W^{k,p}(V)$ for every $V \subset\subset U$.

As we did last time, choose $\eta \in C_c^\infty(U)$ supported in $\overline{B(0,1)}$, positive on $B(0,1)$, and radial with $\int \eta \, dx = 1$. Set

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

Theorem 1. *(Local approximation by smooth functions). Assume* $u \in W^{k,p}(U)$ *for some* $1 \le p < \infty$*, and set*

$$u_\varepsilon = \eta_\varepsilon * u, \qquad \textit{in } U_\varepsilon,$$

where

$$U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Then $u_\varepsilon \in C^\infty(U_\varepsilon)$*, and* $u^\varepsilon \to u \in W_{\text{loc}}^{k,p}(U)$ *as* $\varepsilon \to 0$.

Remark. In fact the more precise statement holds:

$$\|u - u_\varepsilon\|_{W^{k,p}(U_\varepsilon)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Proof.

$$u_\varepsilon(x) = \int_U \eta_\varepsilon(x-y) u(y) \, dy.$$

That $u_\varepsilon \in C^\infty(U_\varepsilon)$ is standard. Indeed, recall

$$|h_{x_i}(x,y)| \le f(y), \qquad |h(x,y)| \le f(y),$$

with

$$\int_U f(y)\,dy \;<\; \infty$$

then

$$D_{x_i}\int_U h(x,y)\,dy \;=\; \int_U h_{x_i}(x,y)\,dy.$$

We want to show that for $V \subset\subset U$, $u_\varepsilon \to u$ in $W^{k,p}(V)$, which is equivalent to showing $D^\alpha u_\varepsilon \to D^\alpha u$ in $L^p(V)$ for $|\alpha| \leq k$.

First we claim that $D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u$ in U^ε.

Indeed, for $x \in U_\varepsilon$, we have

$$D^\alpha u_\varepsilon(x) \;=\; D^\alpha \int_U \eta_\varepsilon(x-y)u(y)\,dy \;=\; (-1)^{|\alpha|}\int_U D_y^\alpha \eta_\varepsilon(x-y)u(y)\,dy \;=\; (\eta_\varepsilon * D^\alpha u)(x).$$

We emphasize that we need $x \in U_\varepsilon$ so that $y \to \eta_\varepsilon(x-y)$ is supported in U. Hence the result follows if we can show

$$(*) \qquad \eta_\varepsilon * u \to u, \qquad \text{in } L^p(V),$$

for we can then apply this with u replaced by $D^\alpha u$ for $|\alpha| \leq k$. To establish (*), we first note that for $x \in U_\varepsilon$,

(**)

$$\begin{aligned} |u_\varepsilon(x)|^p \;=\; \left|\int_U \eta_\varepsilon(x-y)u(y)\,dy\right|^p &\leq \int_U \eta_\varepsilon(x-y)^{(1-1/p)p'}\,dy \int_U \eta_\varepsilon(x-y)|u(y)|^p\,dy \\ &= \int_U \eta_\varepsilon(x-y)|u(y)|^p\,dy \end{aligned}$$

Then

$$\int_{U_\varepsilon} |u_\varepsilon(y)|^p\,dy \;\leq\; \int_{U_\varepsilon}\int_U \eta_\varepsilon(x-y)|u(y)|^p\,dydx \;=\; \|u\|^p_{L^p(U)}.$$

Now for $u, v \in L^p(U)$, setting $v_\varepsilon = \eta_\varepsilon * v$ and applying (**) to the function $u - v$, we get

$$\|u_\varepsilon - v_\varepsilon\|_{L^p(U_\varepsilon)} \;\leq\; \|u - v\|_{L^p(U_\varepsilon)}.$$

Hence

$$\begin{aligned} \|u - u_\varepsilon\|_{L^p(U_\varepsilon)} &\leq \|u - v\|_{L^p(U_\varepsilon)} \;+\; \|v - v_\varepsilon\|_{L^p(U_\varepsilon)} \;+\; \|v_\varepsilon - u_\varepsilon\|_{L^p(U_\varepsilon)} \\ (***) \qquad &\leq 2\|u - v\|_{L^p(U)} \;+\; \|v - v_\varepsilon\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

However, given $\delta > 0$, we can choose $v \in C_c(U)$, so that $\|u - v\|_{L^p(U)} < \delta$. We already showed that $\|v - v_\varepsilon\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$. Hence by choosing ε sufficiently small we get (***)$< 3\delta$.

Theorem 2. If $u \in W^{k,p}(U)$ then there is a sequence of functions $u_j \in C^\infty(U)$ with $u_j \to u$ in $W^{k,p}(U)$.

Proof. Set

$$\begin{aligned} U_i &:= \{x \in U : \operatorname{dist}(x, \partial U) > 1/i\}, \qquad i = 1, 2, \ldots \\ V_i &:= U_{i+3} - \bar{U}_{i+1}, \qquad i = 1, 2, \ldots. \\ W_i &:= U_{i+4} \;-\; \bar{U}_i \;\supset V_i, \qquad i = 1, 2, \ldots. \end{aligned}$$

Choose $V_0 = U_3$ so that

$$U \;=\; \bigcup_{i=0}^{\infty} V_i.$$

By the choice of V_j and W_j we see that each point is contained in at most 4 of the sets W_j.

To be continued.