MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 10. Smooth Approximation.

Partition of Unity: $\eta \in C_c^\infty(U)$ supported in $\overline{B(0,1)}$, positive on $B(0,1)$, and radial with $\int \eta \, dx = 1$. Set

$$\eta_\varepsilon(x) \;=\; \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Last time: **Theorem 1**. $u \in W^{k,p}(U)$ then

$$\|\eta_\varepsilon * u - u\|_{W^{k,p}(U_\varepsilon)} \;\to\; 0, \qquad\qquad \text{as } \varepsilon \to 0.$$

where

$$U_\varepsilon \;=\; \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Theorem 2. If $u \in W^{k,p}(U)$ then there is a sequence of functions $u_j \in C^\infty(U)$ with $u_j \to u$ in $W^{k,p}(U)$.

Last time: we constructed sets $V_j \subset W_j \subset U$ with $V_0 = U_{1/3}$ and

$$U \;=\; \bigcup_{j=0}^\infty V_j,$$

and

$$V_j \subset (W_j)_{\varepsilon_j}, \qquad \varepsilon_j = \frac{1}{(i+3)(i+4)}, \qquad\qquad j = 1, 2, \ldots,$$

and each point of U is contained in at most 4 sets W_j.

Partition of Unity Theorem. If $U \subset \mathbb{R}^n$ is an open set $\{V_i : i = 1, 2, \ldots\}$ is an open cover of U with $V_i \subset U$ for all i, then there exists a collection of functions $\zeta_i \in C^\infty(U)$ such that

$$\operatorname{support} \zeta_i \;\subset\; V_i,$$

and

$$\sum \zeta_i = 1,$$

the sum on the left being locally finite.

Form a partition of unity ζ_j subordinate to V_j. By setting $v_j = \eta_{\varepsilon_j} * (\zeta_j u)$ for ε_j small, we can get $\|v_j - \zeta_j u\|_{W^{k,p}(U)} < \delta/2^{j+1}$, and the support of v_j contained in W_j. Then

$$v \;=\; \sum_{j=0}^\infty v_j$$

is smooth because on each open set $V \subset\subset U$ there are at most finitely many terms in the sum. Moreover, in the $W^{k,p}(U)$ norm we have

$$\|v - u\| \;\le\; \sum_{j=0}^\infty \|v_j - \zeta_j u\| \;<\; \sum_{j=0}^\infty \frac{\delta}{2^{j+1}} \;=\; \delta.$$

Theorem 3. *Suppose that U is bounded and ∂U is C^1. Then for $u \in W^{k,p}(U)$ there exists a sequence $u_j \in C^\infty(\bar{U})$ with $u_j \to u$ in $W^{k,p}(U)$.*

Proof. We will just need to prove this for functions supported on a small neighborhood of a boundary point, because at the end we can take a partition of unity.

For $x = (x_1, \ldots, x_n)$, set $x' = (x_1, \ldots, x_{n-1})$. Suppose $y \in \partial U$. Because ∂U is C^1, we can rotate so that close to y it is given by a graph, that is

$$U \cap B(y,r) = \{x \in B(y,r) : x_n > \gamma(x')\},$$

where $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^1 function. As usual, consider

$$\eta_\varepsilon * u(x) = \int_U \eta_\varepsilon(x-y)u(y)\,dy.$$

Why will this function not approximate u well in $W^{k,p}$? Take for example the function $u = 1$. Then $\eta_\varepsilon * u(x) = 1$ for $x \in U_\varepsilon$. But when $x \notin U_\varepsilon$ then the convolution picks up only part of the integral of η_ε, and indeed $\eta_\varepsilon * u(x) = 0$ when $\operatorname{dist}(x, U) > \varepsilon$. Thus the convolution decreases from 1 to zero in a distance of order ε, so if ε is small, its derivative will be like $1/\varepsilon$ in this set, and the L^1 norm of the derivative will be bounded below.

Geometric Lemma. *Set $V = U \cap B(y, r/2)$ There exists $\lambda > 0$ and $\varepsilon' > 0$ such that $V + \lambda\varepsilon e_n \subset U_\varepsilon$.*

Proof of the Lemma. We will find λ such that when $x \in V$, then $B(x,\varepsilon) + \lambda\varepsilon e_n \subset \{z : z_n > \gamma(z')\}$. By the mean value theorem, if $w \in B(x,\varepsilon)$ and $x \in V$, then setting

$$S = \sup_{B(y',r/2)} \|D\gamma\|,$$

we have

$$\gamma(w') - \gamma(x') \leq S|w' - x'| < S\varepsilon.$$

Now since $x \in V$ we have $x_n > \gamma(x')$ and

$$w_n > x_n - \varepsilon > \gamma(x') - \varepsilon > \gamma(w') - (S+1)\varepsilon.$$

From this we see that setting $\lambda = S + 1$, if $x \in B(y,r)$ then

$$B(x,\varepsilon) + \lambda\varepsilon e_n \subset \{z : z_n > \gamma(z')\}.$$

Now $B(x,\varepsilon) \subset B(y, \varepsilon + r/2)$, and when ε is sufficiently small then $B(y, \varepsilon + r/2) + \lambda\varepsilon e_n \subset B(y,r)$ and the Lemma follows.

Now back to the proof of the Theorem,

Notation. For the function v set

$$v^\varepsilon(x) = v(x + \lambda\varepsilon e_n).$$

We have

$$\begin{aligned}\|(\eta_\varepsilon * u)^\varepsilon - u\|_{W^{k,p}(V)} &\leq \|(\eta_\varepsilon * u)^\varepsilon - u^\varepsilon\|_{W^{k,p}(V)} + \|u^\varepsilon - u\|_{W^{k,p}(V)} \\ &\leq \|\eta_\varepsilon * u - u\|_{W^{k,p}(U_\varepsilon)} + \|u^\varepsilon - u\|_{W^{k,p}(V)}\end{aligned}$$

which converges to zero as $\varepsilon \to 0$. The same holds when u is replaced by $D^\alpha u$ for $|\alpha| \leq k$. Hence $v^\varepsilon \to u$ in $W^{k,p}(V)$.

For the partition of unity argument we followed 5.3.3 steps 4 and 5.