

**Lecture 13. Restriction-Traces.**

**Theorem 1.** *If  $U$  is a bounded open set in  $\mathbb{R}^n$  and  $\partial U$  is  $C^1$ , then there exists a bounded linear operator*

$$T : W^{1,p}(U) \rightarrow L^p(\partial U)$$

*satisfying*

$$(1) \quad Tu = u|_{\partial U}, \quad \text{if } u \in W^{1,p}(U) \cap C(\bar{U}).$$

**Definition.** We call  $Tu$  the *trace* of  $u$  on  $\partial U$ , or the *restriction* of  $u$  to  $\partial U$ .

**Remark.** If we use the Fourier transform to develop the theory of fractional order Sobolev spaces, then one can construct  $T$  satisfying

$$\|Tu\|_{W^{k-1/2,p}(\partial U)} \leq \|u\|_{W^{k,p}(U)},$$

provided the boundary of  $U$  is sufficiently smooth.

**Proof of the Theorem.** First note that we just need to consider functions in  $C^1(\bar{U})$ . Indeed, if we can establish (2) on  $C^1(\bar{U})$ , where  $Tu$  is defined by (1), then  $T$  extends to  $W^{1,p}(U)$  by continuity. To see this, for  $u \in W^{1,p}(U)$ , choose  $u_j \in W^{1,p}(U) \cap C(\bar{U})$  with  $u_j \rightarrow u$ . Then

$$\|Tu_j - Tu_k\|_{L^p(\partial U)} \leq C\|u_j - u_k\|_{W^{1,p}(U)},$$

so  $Tu_j$  is Cauchy in  $L^p(\partial U)$  and hence converges to some limit. We define  $Tu$  to be this limit. It is easy to check that  $T$  is well defined, linear and bounded.

**Model Case.**  $B = B(0, 1)$ ,  $B^+ = B \cap \{x_n > 0\}$ .

$$\Gamma = \{x_n = 0\} \cap B(0, 1/2).$$

**Lemma.** *There exists a constant  $C$  such that if  $u \in C^1(\{x_n \geq 0\})$  then*

$$\|u\|_{L^p(\Gamma)} \leq \|u\|_{W^{1,p}(B^+)}.$$

**Proof.** Set  $x = (x', x_n)$ . Choose  $\zeta \in C_c^\infty(B(0, 1))$  with  $0 \leq \zeta \leq 1$  and  $\zeta = 1$  on  $B(0, 1/2)$ . Now if  $x' \in \Gamma$ , then

$$\begin{aligned} |u(x', 0)|^p &= \left| \int_0^\infty (\zeta u)_{x_n}(x', t) dt \right|^p = \left| \int_0^\infty (\zeta_{x_n} u + \zeta u_{x_n})(x', t) dt \right|^p \\ &\leq 2^{p-1} \left( \int_0^\infty |(\zeta_{x_n} u)(x', t)|^p dt + \int_0^\infty |(\zeta u_{x_n})(x', t)|^p dt \right). \end{aligned}$$

Hence

$$\int_{\Gamma} |u(x', 0)|^p dx' \leq C \left( \int_{B^+} |u|^p dx + \int_{B^+} |u_{x_n}|^p dx \right) \leq C \|u\|_{W^{1,p}(B^+)}^p.$$

Straightening out the boundary: In the case when  $U = \Phi B^+$  where  $\Phi : \bar{B}^+ \rightarrow \bar{U}$  is a  $C^1$  diffeomorphism, and  $\Gamma = \Phi\{(x', 0) : |x'| < 1\}$ , we can prove the analog of the Lemma.

Finally, we cover a neighborhood of the boundary with finitely many sets diffeomorphic to  $B^+$  so that the boundary itself is covered by the image of the sets  $\Gamma$  and take a partition of unity subordinate to this cover as usual.

**Theorem 2.** *If  $U$  is bounded and  $\partial U$  is  $C^1$  and  $u \in W^{1,p}(U)$  then  $u \in W_0^{1,p}(U)$  if and only if  $Tu = 0$  on  $\partial U$ .*

Since  $T$  is bounded, it is easy to see that if  $u \in W_0^{1,p}(U)$ , then  $Tu = 0$ . To show the converse, using the techniques of partitions of unity, straightening out the boundary, and approximating, it is enough to consider the model case where  $U = \mathbb{R}_+^n = \{x : x_n > 0\}$ , and  $u$  is supported in  $\bar{U} \cap \{|x| < 1\}$ . Suppose that  $u$  is  $C^1$  which no assumptions on the trace. Then

$$|u(x', x_n)| \leq |u(x', 0)| + \int_0^{x_n} |u_{x_n}(x', t)| dt.$$

Hence using  $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ , and Hölder's inequality, we get

$$(1) \quad \int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C \left( \int_{\mathbb{R}^{n-1}} |u(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} |u_{x_n}(x', t)|^p dt \right).$$

Now choose  $\zeta \in C^\infty(\mathbb{R})$  with

$$\zeta \equiv 1 \text{ on } [0, 1], \text{ and } \zeta = 0 \text{ on } [2, \infty).$$

Set  $\zeta_m(x) = \zeta(mx)$ , and  $w_m(x) = u(x)(1 - \zeta(x_n))$ .

$$\int_{\mathbb{R}_+^n} |w_m - u|^p dx = \int_{\mathbb{R}_+^n} |u|^p |\zeta(x_n)|^p dx \leq \int_{x_n=0}^{2/m} \int_{|x'| < 1} |u(x', x_n)|^p dx' dx_n \rightarrow 0$$

as  $m \rightarrow \infty$ . Similarly,

$$(2) \quad \int_{\mathbb{R}_+^n} |Dw_m - Du|^p dx \leq C \int_{\mathbb{R}_+^n} |\zeta_n|^p |Du|^p dx + Cm^p \int_0^{2/m} \int_{|x'| < 1} |u(x', t)|^p dx' dt.$$

The first term on the right in (2) is bounded by

$$\int_{x_n=0}^{2/m} \int_{|x'| < 1} |Du(x', x_n)|^p dx' dx_n \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

**Lemma.** *If  $Tu = 0$  on  $\{x_n = 0\}$  then*

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dt.$$

Hence the second term in (2) is bounded by

$$\begin{aligned} C m^p \int_0^{2/m} t^{p-1} \int_0^t \int_{\mathbb{R}^{n-1}} |Du(x', s)|^p dx' ds dt \\ \leq C \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du(x', x_n)|^p dx' dx_n \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$