MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 15. Sobolev Inequalities.

Last time: The G-N-S inequality states that if $u \in C_c^1(\mathbb{R}^n)$ then

$$(*) \qquad \|u\|_{L^{p^*}(\mathbb{R}^n)} \;\le\; C\|Du\|_{L^p(\mathbb{R}^n)}.$$

Here,

$$\frac{1}{p^*} \;=\; \frac{1}{p} \;-\; \frac{1}{n}.$$

We proved the case $p = 1$. For $1 < p < n$, we apply the case $p = 1$ to $|u|^\gamma$ for $\gamma > 1$ to be seleected to get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\gamma n/(n-1)}\right)^{n/(n-1)} &\le \int_{\mathbb{R}^n} |D|u|^\gamma|\,dx \;=\; \gamma\int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\,dx \\ &\le \gamma\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)p/(p-1)}\right)^{(p-1)/p}\left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{1/p}.\end{aligned}$$

Choosing γ so that

$$(*) \qquad \frac{\gamma n}{n-1} \;=\; (\gamma-1)\frac{p}{p-1},$$

by setting

$$\gamma \;=\; \frac{p(n-1)}{n-p} \;>\; 1,$$

we see that (*) equals p^*, and we get the result.

Corollary 1. *If $u \in W^{1,p}(\mathbb{R}^n)$ then $u \in L^{p^*}(\mathbb{R}^n)$ and u satisfies (*).*

Proof. Let $u \in W^{1,p}(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, there exists a sequence $u_j \in C_c^\infty(\mathbb{R}^n)$ with $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$. Then from G-N-S (*), u_j is Cauchy in $L^{p^*}(\mathbb{R}^n)$ so $u_j \to u^*$ in $L^{p^*}(\mathbb{R}^n)$ for some u^*. But then $u = u^*$.

Corollary 2. *Let U be a bounded open subset of $\mathbb{R}^n$. Suppose that ∂U is C^1. Suppose $1 \le p < n$.*
(a). $u \in W^{1,p}(\mathbb{R}^n)$ then $u \in L^{p^*}(\mathbb{R}^n)$

$$\|u\|_{L^{p^*}(U)} \;\le\; C\|u\|_{W^{1,p}(U)}.$$

(b). If $u \in W_0^{1,p}(U)$ then

$$\|u\|_{L^{p^*}(U)} \;\le\; C\|Du\|_{L^p(U)}.$$

Proof. (a). Let $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ be a bounded extension operator. Then we have

$$\|u\|_{L^{p^*}(U)} \;\le\; \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \;\le\; C\|D(Eu)\|_{L^p(\mathbb{R}^n)} \;\le\; C'\|u\|_{W^{1,p}(U)}.$$

(b). For functions v defined on U, set

$$\tilde{v} \;=\; \begin{cases} v & \text{on } U, \\ 0 & \text{on } \mathbb{R}^n \setminus U. \end{cases}$$

Then we claim that if $u \in W_0^{1,p}(U)$ then

$$(**) \qquad \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|u\|_{W^{1,p}(U)}.$$

Now u is a limit in $W^{1,p}(U)$ of a sequence of functions $u_j \in C_c^\infty(U)$. But then $\tilde{u}_j$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$ and so $\tilde{u}_j \to u^*$ for some function $u^* \in W^{1,p}(\mathbb{R}^n)$. Since $\tilde{u}_j \to \tilde{u}$ in $L^p(\mathbb{R}^n)$ we get $u^* = \tilde{u}$. When u is replaced by u_j, (**) holds, and hence it holds for u.

$$\|u\|_{L^{p^*}(U)} \;\le\; \|\tilde{u}\|_{L^{p^*}(\mathbb{R}^n)} \;\le\; C\|D(\tilde{u})\|_{L^p(\mathbb{R}^n)} \;=\; C\|Du\|_{L^p(U)}.$$

Corollary. *If U is open and bounded with C^1 boundary, then $\|u\|_{W^{1,p}(U)}$ and $\|Du\|_{L^p(\mathbb{R}^n)}$ give equivalent norms on $W_0^{1,p}(U)$.*

Proof. By Hölder's inequality, for $u \in W_0^{1,p}(\mathbb{R}^n)$,

$$\|u\|_{L^p(U)} \;\le\; (\text{Volume}(U))^{1/(p^*/p)'}\|u\|_{L^{p^*}(U)} \;\le\; C\|Du\|_{L^p(U)}.$$

Hölder Spaces. Suppose U is open in $\mathbb{R}^n$ and $0 < \gamma \le 1$. If $u : U \to \mathbb{R}$ is bounded and continuous, then

$$\|u\|_{C(U)} \;=\; \sup_{x\in U} |u(x)|,$$

and the γ^{th}-Hölder norm of $u : U \to \mathbb{R}^n$ is

$$\|u\|_{C^{0,\gamma}(U)} \;=\; \sup_{x\in U} |u(x)| \;+\; \sup_{\substack{x,y\in U \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

The Hölder space $C^{k,\gamma}(U)$ is the space of those functions $u \in C^k(U)$ for which the norm

$$\|u\|_{C^{k,\gamma}(U)} \;=\; \sum_{|\alpha|<k} \|D^\alpha u\|_{C(U)} \;+\; \sum_{|\alpha|=k} \|u\|_{C^{0,\gamma}(U)}$$

is finite. It is a Banach space.

Hölder Spaces. If $U \subset \mathbb{R}^n$ is open and $u \in C(\bar{U})$, the γth Hölder seminorm of u is

$$[u]_{C^{0,\gamma}(U)} \;:=\; \sup_{x,y\in U,\, x\neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}.$$

The γth Hölder norm is

$$\|u\|_{C^{0,\gamma}(U)} \;=\; \|u\|_{C(U)} \;+\; [u]_{C^{0,\gamma}(U)}.$$

The γth Hölder space is the space of functions u for which this norm is finite. It is a Banach space. We remark that $C^{0,\gamma}((-1,1))$ contains the function $|x|^\gamma$.

Morrey's Inequality. *If $p > n$ then there exists $C = C(p, n)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \;\le\; C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$ where

$$\gamma := 1 - n/p.$$

Remark. We note that the value of γ fits into a general principle for Sobolev inequalities. If $p^{(r)}$ is defined by

$$\frac{1}{p^{(r)}} \;=\; \frac{1}{p} - \frac{r}{n},$$

then $u \in W^{k,p}(\mathbb{R}^n)$ implies $u \in W^{k-r,p^{(r)}}(\mathbb{R}^n)$. Now if $p^{(r)} = \infty$, then $r = n/p$. Heuristically then, if u has 1 derivative in L^p, you can subtract off n/p derivatives to get a bounded function. However, you then still have $1 - n/p$ derivatives left, and we can interpret this heuristically $u \in C^{0,1-n/p}(\mathbb{R}^n)$.

Proof of Morrey's inequality. We first show that

$$⨍_{B(x,r)} |u(y) - u(x)|\, dy \;\le\; C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

This is easy to prove. Indeed, if $w \in \partial B(0,1)$ then in polar coordinates we are trying to show that

(*)

$$\frac{C}{r^n}\int_{t=0}^{r}\int_{S^{n-1}} |u(x+tw)-u(x)|\, dS(w)\, t^{n-1}dt \;\le\; \int_{s=0}^{r}\int_{S^{n-1}} |Du|(x+sw)\, dS(w)\, ds.$$

To see this, we have

$$|u(x+tw) - u(x)| \;=\; \left|\int_0^t \frac{d}{ds}u(x+sw)\, ds\right| \;\le\; \int_0^t |Du|(x+sw)\, ds.$$

Hence the left hand side of (*) is bounded by

$$\begin{aligned}\int_{t=0}^{r}\int_{s=0}^{t}\int_{S^{n-1}} |Du|(x+sw)\, dS(w)\, t^{n-1}\, dsdt \;&=\; \frac{1}{n}\int_{s=0}^{r}\int_{S^{n-1}} |Du|(x+sw)\, dS^{n-1}(r^n-s^n)\, ds\\ &\le\; \frac{r^n}{n}\int_{s=0}^{r}\int_{S^{n-1}} |Du|(x+sw)\, dS^{n-1}(r^n-s^n)\, ds.\end{aligned}$$

Here we have used Fubini and computed

$$\int_{t=s}^{r} t^{n-1}\, dt \;=\; \frac{r^n - s^n}{n}.$$